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Solution to the initial value problem for a high-gain FEL via Van Kampen's method ☆

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Abstract

Using Van Kampen's normal mode expansion, we solve the initial value problem for a high-gain free-electron laser described by the three-dimensional Maxwell–Klimontovich equations. An expression of the radiation spectrum is given for the process of coherent amplification and self-amplified spontaneous emission. It is noted that the input coupling coefficient for either process increases with the initial beam energy spread. The effective start-up noise is identified as the coherent fraction of the spontaneous undulator radiation in one field gain length, and is larger with increasing energy spread and emittance mainly because of the increase in gain length. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In a high-gain free-electron laser (FEL), a coherent external signal or the incoherent undulator radiation can initiate the FEL interaction for an exponentially growing coherent radiation. Such a radiation is a promising source for future-generation X-ray facilities. Thus, it is important to understand how the exponential process starts and how the incoherent radiation develops into a coherent source.

The FEL initial value problem was solved in one-dimensional (1-D) theory [1,2] using the Laplace transform technique. The three-dimensional (3-D) initial value problem for a parallel beam was studied by Van Kampen's method in Ref. [3] and by a Green's function technique in Refs. [4,5]. Extension of Van Kampen's method to include the emittance effect was made in Refs. [6,7]. Using an equivalent method, Xie [8] independently obtained the solution to the initial value problem including emittance and numerically found that the effective start-up noise in self-amplified spontaneous emission (SASE) becomes significantly larger with finite emittance and energy spread.

Inspired by the work of Xie, we explain the solution to the FEL initial value problem using Van Kampen's method applicable to the 3-D case including betatron focusing and emittance. We then attempt to provide an understanding of the dependence of the effective start-up noise on beam parameters. Two factors determining the start-up

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process are identified. The input coupling coefficient for both coherent amplification (CA) and SASE is found to increase with the initial energy spread. The effective start-up noise is shown to be the coherent fraction of the spontaneous undulator radiation in the first field gain length, generalizing the result of Ref. [5] for a beam with vanishing energy spread and emittance. The effective start-up noise appears to be larger with increasing energy spread and emittance mainly because of the increase of the gain length. Fluctuation in initial electron velocities (due to beam energy spread and angular spread) do not seem to contribute to any additional start-up noise.

2. The dynamic equations and the initial conditions

We follow closely to the notations of Ref. [7], which makes extensively use of the FEL parameter ρ [9] to scale all the dynamical variables. Assuming that the initial smooth electron distribution f_0 is matched to the undulator channel transversely and has a uniform longitudinal profile, the Maxwell–Klimontovich equations in the small signal regime can be written as [7]

$$\left(\frac{\partial}{\partial \bar{z}} - \mathrm{i}\mathbf{M}\right)\Phi(\bar{z}) = 0 \tag{1}$$

where the state vector is

$$\Phi(\bar{z}) = \begin{pmatrix} a_\nu(\bar{\boldsymbol{x}};\bar{z}) \\ f_\nu(\bar{\eta},\bar{\boldsymbol{x}},\bar{\boldsymbol{p}};\bar{z}) \end{pmatrix} \tag{2}$$

and the operator $\mathbf{M}$ is defined through

$$\mathbf{M}\Phi(\bar{z}) = \begin{pmatrix} (-\bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2})a_\nu - \mathrm{i}\int \mathrm{d}^2\bar{p}\int \mathrm{d}\bar{\eta} f_\nu \\ -\mathrm{i}a_\nu \frac{\partial f_0}{\partial \bar{\eta}} + [-\nu(\bar{\eta} - \frac{1}{2}(\bar{\boldsymbol{p}}^2 + \bar{k}^2\bar{\boldsymbol{x}}^2)) \\ +\mathrm{i}(\bar{\boldsymbol{p}}\frac{\partial}{\partial \bar{\boldsymbol{x}}} - \bar{k}_\beta^2\bar{\boldsymbol{x}}\frac{\partial}{\partial \bar{\boldsymbol{p}}})]f_\nu \end{pmatrix}. \tag{3}$$

Here the scaled undulator distance $\bar{z}$ is the independent "time" variable, a_ν and f_ν are the νth fourier component of the scaled electric field and the perturbed electron distribution function f_1, respectively, $(\theta, \bar{\eta}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{p}})$ are the longitudinal and the transverse phase space variables, $\bar{\boldsymbol{\nabla}}_\perp = \partial/(\partial\bar{\boldsymbol{x}})$ is the scaled transverse Laplacian, $\bar{\nu} = (\nu - 1)/(2\rho)$, $\bar{k}$ is the scaled natural focusing strength, and $\bar{k}_\beta$ is the total scaled focusing strength (including the natural focusing and the average effects of the external focusing).

The evolution of the radiation field and the distribution function in the start-up and the exponential growth regimes is completely determined by Eq. (1) and the initial value $\Phi(0)$ of the state vector. The latter is specified by the external signal $a_\nu(0)$ and the shot noise $f_\nu(0) = \int(2\rho\,\mathrm{d}\theta/2\pi)\mathrm{e}^{-\mathrm{i}\nu\theta}f_1(0)$. Although the ensemble average of $f_\nu(0)$ is zero, physically meaningful quantities such as intensity can be computed by using the relation [10] (see also Ichimaru [10]):

$$\langle f_\nu(\bar{\eta},\bar{\boldsymbol{x}},\bar{\boldsymbol{p}};0)f_\nu(\bar{\eta}',\bar{\boldsymbol{x}}',\bar{\boldsymbol{p}}';0)\rangle = \frac{2k_1^2k_\mathrm{u}\rho^3\theta_\mathrm{b}}{\pi^2 n_0}\delta(\bar{\eta}-\bar{\eta}')\delta(\bar{\boldsymbol{x}}-\bar{\boldsymbol{x}}')\delta(\bar{\boldsymbol{p}}-\bar{\boldsymbol{p}}')f_0. \tag{4}$$

Here $\lambda_1 = 2\pi/k_1$ is the resonant radiation wavelength, $\lambda_\mathrm{u} = 2\pi/k_\mathrm{u}$ is the undulator period, n_0 is the peak electron volume density, and θ_b is the bunch length in units of $2\pi/\lambda_1$.

3. Van Kampen's normal mode expansion

The initial value problem formulated in the previous section can be solved by expanding the solution in terms of the eigenvectors of Eq. (1). The coefficients of the expansion are determined from the initial conditions if the eigenvectors are mutually orthogonal under a suitably defined scalar product. The procedure is well-known in quantum mechanics in which all operators are Hermitian. Here $\mathbf{M}$ is not a Hermitian operator, and we employ the extension of the method developed by Van Kampen [11] in studying the 1-D plasma waves.

Let us first find the eigenvalues and the eigenvectors of Eq. (1), defined to be solutions

$$\mathrm{e}^{-\mathrm{i}\mu_n\bar{z}}\Psi_n = \mathrm{e}^{-\mathrm{i}\mu_n\bar{z}}\begin{pmatrix} A_n(\bar{\boldsymbol{x}}) \\ F_n(\bar{\eta},\bar{\boldsymbol{x}},\bar{\boldsymbol{p}}) \end{pmatrix}. \tag{5}$$

Solving the eigenvalue equation $(\mu_n + \mathbf{M})\Psi_n = 0$, we obtain the mode equation

$$\left(-\mathrm{i}\mu_n + \mathrm{i}\bar{\nu} + \frac{\bar{\boldsymbol{\nabla}}_\perp^2}{2\mathrm{i}}\right) A_n(\bar{\boldsymbol{x}}) - \int \mathrm{d}^2\bar{p} \int \mathrm{d}\bar{\eta} \int_{-\infty}^{0} \mathrm{d}\tau A_n(\bar{\boldsymbol{x}}_\beta(\tau)) \mathrm{e}^{\mathrm{i}\phi_\beta(\tau)} \frac{\partial f_0}{\partial \bar{\eta}} = 0 \quad (6)$$

where

$$\bar{\boldsymbol{x}}_\beta(\tau) = \bar{\boldsymbol{x}} \cos \bar{k}_\beta \tau + \frac{\bar{\boldsymbol{p}}}{\bar{k}_\beta} \sin \bar{k}_\beta \tau$$

$$\phi_\beta(\tau) = [\bar{\eta} - \tfrac{1}{2}(\bar{\boldsymbol{p}}^2 + \bar{k}_\beta^2 \bar{\boldsymbol{x}}^2) - \mu_n]\tau. \quad (7)$$

Eq. (6) is the dispersion relation derived in Refs. [3,12] for natural focusing only (i.e., $\bar{k}_\beta = \bar{k}$). For alternating-gradient focusing with $\bar{k}_\beta \gg \bar{k}$, it is shown to be also valid after averaging properly over many periods of the focusing structure [13,14]. It can be solved using a variational principle [12,15] and a matrix formalism [15]. In general, a discrete set of eigenvalues and eigenmodes exists.

Van Kampen orthogonality of these eigenvectors is constructed by introducing the adjoint eigenvalue equation $(\tilde{\mu}_n + \tilde{\mathbf{M}})\tilde{\Psi}_n = 0$, where $\tilde{\mu}_n$ and $\tilde{\Psi}_n$ are the adjoint eigenvalues and eigenvectors of the adjoint operator $\tilde{\mathbf{M}}$. The formal procedure can be found in Ref. [7]. In the high-gain regime where the fundamental mode $A_0(\bar{\boldsymbol{x}})$ dominates because its eigenvalue μ_0 has the largest imaginary part μ_I, we project the initial conditions to this mode à la Van Kampen and obtain the evolution of the electric field

$$a_\nu(\bar{\boldsymbol{x}}; \bar{z}) = \frac{A_0(\bar{\boldsymbol{x}})\mathrm{e}^{-\mathrm{i}\mu_0 \bar{z}}}{C_0} \left[\int \mathrm{d}^2\bar{x}' A_0(\bar{\boldsymbol{x}}') a_\nu(\bar{\boldsymbol{x}}'; 0) + \int \mathrm{d}^2\bar{x}' \int \mathrm{d}^2\bar{p} \int \mathrm{d}\bar{\eta} f_\nu(\bar{\eta}, \bar{\boldsymbol{x}}', \bar{\boldsymbol{p}}; 0) \times \int_{-\infty}^{0} \mathrm{d}\tau A_0(\bar{\boldsymbol{x}}_\beta(\tau)) \mathrm{e}^{\mathrm{i}\phi_\beta(\tau)} \right] \quad (8)$$

where

$$C_0 = \int \mathrm{d}^2\bar{x} A_0^2(\bar{\boldsymbol{x}}) + \int \mathrm{d}^2\bar{x} \int \mathrm{d}^2\bar{p} \int \mathrm{d}\bar{\eta} \frac{\partial f_0}{\partial \bar{\eta}} \times \left[\int_{-\infty}^{0} \mathrm{d}\tau A_0(\bar{\boldsymbol{x}}_\beta(\tau)) \mathrm{e}^{\mathrm{i}\phi_\beta(\tau)} \right]^2. \quad (9)$$

These expressions have been obtained independently by Xie [8] using an equivalent method. The first term in the square bracket of Eq. (8) describes the process of coherent amplification, which starts from an external signal $a_\nu(0)$. The second term describes the process of self-amplified spontaneous emission, which starts from white noise. Eq. (8) for the parallel e-beam (with vanishing emittance) reduces to those of Refs. [3,4]. The ensemble averaged spectrum of the radiation intensity (power per unit area) can be computed with the help of Eq. (4):

$$\begin{aligned} \frac{1}{\rho I_\mathrm{beam}} \frac{\mathrm{d}I}{\mathrm{d}\nu} &= \frac{2\pi}{(2\rho)^2 \theta_\mathrm{b}} \langle |a_\nu(\bar{\boldsymbol{x}}; z)|^2 \rangle \\ &= \frac{1}{|C_0|^2} |A_0(\bar{\boldsymbol{x}})|^2 \mathrm{e}^{2\mu_\mathrm{I} \bar{z}} \\ &\quad \times \left[\frac{2\pi}{(2\rho)^2 \theta_\mathrm{b}} \left| \int \mathrm{d}^2\bar{x}' A_0(\bar{\boldsymbol{x}}') a_\nu(\bar{\boldsymbol{x}}'; 0) \right|^2 \right. \\ &\quad + \frac{2k_1^2 k_\mathrm{u} \rho}{2\pi n_0} \int \mathrm{d}^2\bar{x}' \int \mathrm{d}^2\bar{p} \int \mathrm{d}\bar{\eta} f_0(\bar{\eta}, \bar{\boldsymbol{x}}', \bar{\boldsymbol{p}}) \\ &\quad \left. \times \left| \int_{-\infty}^{0} \mathrm{d}\tau A_0(\bar{\boldsymbol{x}}_\beta(\tau)) \mathrm{e}^{\mathrm{i}\phi_\beta(\tau)} \right|^2 \right] \end{aligned} \quad (10)$$

where $I_\mathrm{beam} = \gamma_0 mc^3 n_0$, and $\gamma_0 mc^2$ is the beam energy.

4. Effects of energy spread

To isolate the energy spread effects in the FEL start-up process, we look at the 1-D limit of the above results by setting $A_n(\bar{\boldsymbol{x}}) = 1$, $\int \mathrm{d}^2\bar{x} = 2k_1 k_\mathrm{u} \rho \Sigma$ (Σ is the beam cross-section) and dropping $\int \mathrm{d}^2\bar{p}$ and the transverse Laplacian. The mode Eq. (6) reduces to the 1-D dispersion relation [1]:

$$D(\mu) = \mu - \bar{\nu} - \int \mathrm{d}\bar{\eta} \frac{\mathrm{d}V/\mathrm{d}\bar{\eta}}{\bar{\eta} - \mu} = 0 \quad (11)$$

where $f_0 = V(\bar{\eta})$ with $\int \mathrm{d}\bar{\eta} V(\bar{\eta}) = 1$. For a mono-energetic beam (i.e., $V(\bar{\eta}) = \delta(\bar{\eta})$), this reduces to the cubic equation [9] with a growing, a decaying and an oscillatory solution. The intensity spectrum of Eq. (10) becomes the power spectrum of

Ref. [1]:

$$\frac{dP}{d\omega} = \frac{dI}{d\nu}\frac{\Sigma}{ck_1} = g_A e^{2\mu_I \bar{z}}\left[\left(\frac{dP}{d\omega}\right)_C + \left(\frac{dP}{d\omega}\right)_S\right], \tag{12}$$

where

$$g_A = \left[1 - 2\int d\bar{\eta}\frac{V(\bar{\eta})}{(\bar{\eta}-\mu)^3}\right]^{-2} = \left(\frac{dD}{d\mu}\right)^{-2}$$

$$\left(\frac{dP}{d\omega}\right)_C = \frac{\pi P_{beam}}{2\rho c k_1 \theta_b}|a_\nu(0)|^2 \quad \text{(coherent source)}$$

$$\left(\frac{dP}{d\omega}\right)_S = g_S\frac{\rho\gamma_0 mc^2}{2\pi} \quad \text{(start-up noise)}$$

$$g_S = \int d\bar{\eta}\frac{V(\bar{\eta})}{|\bar{\eta}-\mu|^2} = \int d\bar{\eta}\frac{V(\bar{\eta})}{\mu_I^2 + (\bar{\eta}-\mu_R)^2}. \tag{13}$$

Here $P_{beam} = I_{beam}\Sigma$ is the beam power, and $\mu = \mu_R + i\mu_I$ is a function of the frequency detuning $\bar{\nu}$ through the dispersion relation. For CA, the amplification occurs at the frequency defined by the frequency of the coherent source. For SASE, the frequency dependence is determined by $\mu_I(\bar{\nu})$ in the exponent of Eq. (12). Thus, g_A and g_S, evaluated at the optimal detuning $\bar{\nu}_0$ where the growth rate μ_I reaches the maximum, determine the input coupling to the exponentially growing mode and the effective start-up noise in units of $\rho\gamma_0 mc^2/(2\pi)$, respectively.

In Ref. [1], $G = g_A g_S$ has been computed numerically for a flat-top energy distribution and has been found to increase initially with energy spread. For a Gaussian energy distribution

$$V(\bar{\eta}) = \frac{1}{\sqrt{2\pi}\bar{\sigma}_\eta}\exp\left(-\frac{\bar{\eta}^2}{2\bar{\sigma}_\eta^2}\right) \tag{14}$$

we compute μ_I, g_A and g_S as functions of the r.m.s energy spread $\bar{\sigma}_\eta = \sigma_\gamma/(\gamma_0\rho)$ (see Fig. 1) and find that both g_A and g_S increase with $\bar{\sigma}_\eta$. For a monoenergetic beam, any initial signal (external or spontaneous) couples equally well to the three (growing, decaying and oscillatory) modes that have the same normalization factor, hence we have the well-known $g_A = \frac{1}{9}$. However, g_A is larger for a larger energy spread, approaching $\frac{1}{4}$ for the flat-top model and 1 for the Gaussian model. We note that the increase of the input coupling coefficient g_A is the same for both CA and SASE, and a plausible explanation may be made [16] as follows: an electron beam with larger energy spread is less sensitive to the detuning effect due to the deceleration of the electrons caused by the FEL interaction. Thus, it can couple more effectively with the exponentially growing radiation.

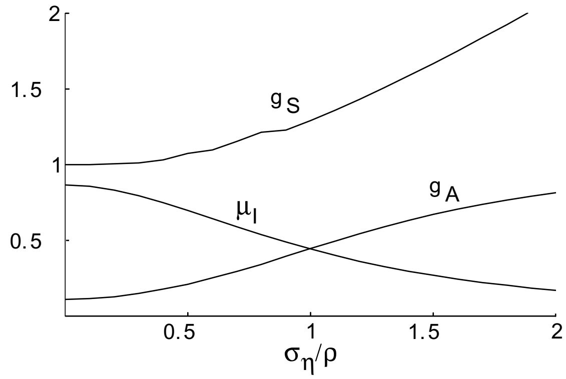

Fig. 1. The behavior of μ_I, g_A and g_S as functions of the r.m.s. energy spread $\bar{\sigma}_\eta = \sigma_\eta/\rho$ for a Gaussian energy distribution.

The increase in the effective start-up noise through g_S may be interpreted in the following way. First of all, $g_S = 1$ for a monoenergetic beam, and the quantity $\rho\gamma_0 mc^2/(2\pi)$ is approximately the spontaneous undulator radiation in the first field gain length $L_{g0} = 1/(k_u\rho\sqrt{3})$ [5]. For a beam with a finite energy spread, the spontaneous radiation spectrum in the forward direction is the convolution of the beam energy spectrum and the undulator radiation spectrum with an intrinsic bandwidth $2\Delta\eta = \Delta\nu = \Delta\omega/\omega \sim 2\pi/(k_u z)$. After the first field gain length $z = L_g = (2k_u\rho\mu_I)^{-1}$, the spontaneous radiation spectrum becomes

$$\left(\frac{dP}{d\omega}\right)_{L_g}^{spont} = \frac{\rho\gamma_0 mc^2}{2\pi\mu_I^2}\int d\bar{\eta}V(\bar{\eta})S_u\left(\frac{\bar{\eta}-\bar{\nu}}{2\mu_I}\right) \tag{15}$$

where $S_u(x) = \sin^2(x)/x^2$ is the undulator spectral function [18]. Rewriting Eq. (13) as

$$\left(\frac{dP}{d\omega}\right)_S = \frac{\rho\gamma_0 mc^2}{2\pi\mu_I^2}\int d\bar{\eta}V(\bar{\eta})\left[\frac{1}{1+(\bar{\eta}-\mu_R)^2/\mu_I^2}\right] \tag{16}$$

and comparing with Eq. (15), we may interpret the effective start-up noise as the fraction of the spontaneous undulator radiation in the first field gain length within the coherent gain bandwidth $\Delta\bar{\eta} \sim \Delta\bar{\nu} \sim \mu_I$ (much narrower than the intrinsic undulator bandwidth $2\pi\mu_I$). Using the Gaussian energy distribution in Eq. (14) and approximating the Lorentzian in the square bracket of Eq. (16) by another Gaussian, we can carry out the $\bar{\eta}$-integral to obtain

$$\left(\frac{dP}{d\omega}\right)_S \approx \frac{\rho\gamma_0 mc^2}{2\pi\mu_I^2}\exp\left(-\frac{\mu_R^2}{2\bar{\sigma}_\eta^2+\mu_I^2}\right)\frac{1}{\sqrt{1+2\bar{\sigma}_\eta^2/\mu_I^2}}. \tag{17}$$

With increasing energy spread, the coherent fraction of the spontaneous radiation decreases, but the drop in the growth rate significantly increases the spontaneous radiation power in one field gain length, leading to the overall increase of the effective start-up noise through g_S (as seen in Fig. 1). In fact, for large values of the energy spread $\bar{\sigma}_\eta^2 \gg 1$, $\mu_R \approx \bar{\nu}_0 \approx -\bar{\sigma}_\eta$ and $\mu_I \approx 0.76/\bar{\sigma}_\eta^2$ [17], so that $(dP/d\omega)_S \propto \bar{\sigma}_\eta$ increases without bound because the noise required to start the SASE process is infinite.

5. Effects of emittance

We now return to the full 3-D Eq. (10) and consider SASE (the second term) only. Assuming that the betatron oscillations are slow on the scale of the gain length, we take $\bar{k}_\beta\tau \ll 1$, $\bar{\mathbf{x}}_\beta(\tau) \approx \bar{\mathbf{x}}'$ and integrate $\int dx^2(dI/d\omega)$ to obtain the SASE power spectrum

$$\left(\frac{dP}{d\omega}\right)_{SASE} \approx g_A^{3D}\left(\frac{dP}{d\omega}\right)_S e^{2\mu_I\bar{z}}. \tag{18}$$

Here $g_A^{3D} = \int d^2\bar{x}|A_0(\bar{\mathbf{x}})|^2/|C_0|^2$ is the input coupling coefficient. The effective start-up noise is

$$\left(\frac{dP}{d\omega}\right)_S = \frac{\rho\gamma_0 mc^2}{2\pi\mu_I^2}\int d^2\bar{x}|A_0(\bar{\mathbf{x}})|^2\int d^2\bar{p}\,U(\bar{\mathbf{p}}^2+\bar{k}_\beta^2\bar{\mathbf{x}}^2) \times d\bar{\eta}\,V(\bar{\eta})\left[1+\left(\frac{\bar{\eta}-(\bar{\mathbf{p}}^2+\bar{k}_\beta^2\bar{\mathbf{x}}^2)/2-\mu_R}{\mu_I}\right)^2\right]^{-1} \tag{19}$$

where $U(\bar{\mathbf{p}}^2+\bar{k}_\beta^2\bar{\mathbf{x}}^2)$ is the electron transverse distribution function. Eq. (19) provides a similar phase-space convolution as the spontaneous undulator radiation when the effects of electron angular spread is taken into account [18], except that the spectral function is a Lorentzian instead of the undulator spectrum S_u at one field gain length. Identifying μ_I as the bandwidth of $\bar{\eta}$ (or $\bar{\nu}$) as in Section 4 and $\sqrt{\mu_I}$ as the angular spread of the fundamental mode (or $L_g = (2k_u\mu_I\rho)^{-1}$ as the Rayleigh length), we may interpret the effective start-up noise as the phase-space convolution of the spontaneous undulator radiation in the first field gain length with the coherent fundamental laser mode.

For numerical computation, we approximate $A_0(\bar{\mathbf{x}}) = \exp(-w|\bar{\mathbf{x}}|^2/\bar{\sigma}_x^2)$, where $w = w_R + w_I$ is a complex number characterizing the fundamental radiation mode, and $\bar{\sigma}_x$ is the scaled transverse e-beam size and is related to the r.m.s. emittance $\varepsilon = \bar{\sigma}_x^2\bar{k}_\beta/k_1$. Eq. (18) can be written as

$$\left(\frac{dP}{d\omega}\right)_{SASE} \approx g_A^{3D}g_S^{3D}\frac{\rho\gamma_0 mc^2}{2\pi}e^{2\mu_I\bar{z}} \tag{20}$$

where

$$g_A^{3D} \approx \left|1-4iw\int_{-\infty}^0 d\tau_1\int_{-\infty}^0 d\tau_2 \times \frac{(\tau_1+\tau_2)\exp[-(\bar{\sigma}_\eta^2/2)(\tau_1+\tau_2)^2-i\mu_0(\tau_1+\tau_2)]}{[1+i\bar{k}_\beta^2\bar{\sigma}_x^2(\tau_1+\tau_2)][1+4w+i\bar{k}_\beta^2\bar{\sigma}_x^2(\tau_1+\tau_2)]}\right|^{-2}$$

$$g_S^{3D} \approx \frac{4|w|^2}{w_R}\int_{-\infty}^0 d\tau_1\int_{-\infty}^0 d\tau_2 \times \frac{\exp[-(\bar{\sigma}_\eta^2/2)(\tau_1-\tau_2)^2-i(\mu_0\tau_1-\mu_0^*\tau_2)]}{[1+i\bar{k}_\beta^2\bar{\sigma}_x^2(\tau_1-\tau_2)][1+4w_R+i\bar{k}_\beta^2\bar{\sigma}_x^2(\tau_1-\tau_2)]}. \tag{21}$$

For example, using the current design parameters of the Linac Coherent Light Source (LCLS) [19], we have $\bar{\sigma}_r = 2.8$, $\bar{\sigma}_\eta = 0.45$, and $\bar{k}_\beta = 0.29$. The fundamental guided mode has a complex growth rate $\mu_0 = -1.2 + 0.42\mathrm{i}$ and a mode profile determined by $w = 0.64 - 0.50\mathrm{i}$ at the optimal detuning $\bar{\nu}_0 = -1.0$ [7]. Hence we obtain $g_{\mathrm{A}}^{3\mathrm{D}} \approx 0.3$ and $g_{\mathrm{S}}^{3\mathrm{D}} \approx 2.6$, both larger than the values with vanishing energy spread and emittance by a factor of ~ 3 each.

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