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Linear theory of transverse and longitudinal ionization cooling in a quadrupole channel ☆

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Abstract

Ionization cooling is the best-known cooling mechanism for the envisioned muon colliders and neutrino factories. In this paper, using the moment-equation approach, we present a linear theory of ionization cooling dynamics in 6D phase space in a quadrupole focusing channel. A simple set of differential equations that governs the evolution of both the transverse and longitudinal emittances is derived, and closed-form solutions are given. Two new significant heating processes have been identified. This theory is analogous to the standard linear theory in electron storage rings. Multiple scattering integrals and energy straggling integrals, quantities like the synchrotron radiation integrals, are introduced to specify the cooling process and the equilibrium emittances in a periodic cooling channel.

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1. Introduction

Ionization cooling channels are being developed to reduce the transverse and longitudinal emittances of a muon beam for envisioned neutrino factories and muon colliders [1–4]. Ionization cooling is achieved by reducing the muons' momenta through ionization energy loss in absorbers and replenishing the momentum loss only in the longitudinal direction through RF cavities. This mechanism can effectively reduce the transverse emittance of a muon beam in the same way as radiation damping does to an electron beam. To obtain longitudinal cooling, dispersion is introduced to spatially separate muons of different longitudinal momenta, and a wedged absorber is used to reduce the momentum spread. Such a longitudinal cooling scheme is called "emittance exchange" because the longitudinal cooling is achieved at the expense of transverse heating or a reduced transverse cooling rate.

Ionization cooling in a quadrupole channel has been discussed extensively by many authors, especially Neuffer [4,5], for both transverse cooling and longitudinal cooling through emittance

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exchange. Although the basic physics is well understood, a consistent 6D theory analogous to electron ring theory has been lacking. In this paper, we present such a linear theory for 6D ionization cooling of a matched beam using the moment-equation approach. We adopt this approach because: (a) the second moments of beam phase-space distribution contain most of the important beam properties such as rms beam size and angular divergence, (b) the evolution equations of the second moments close on themselves for linear dynamics that often dominate beam evolution, and (c) it is equivalent to solving the linearized Fokker–Planck equation. There are 21 independent second moments for 6D phase space. In general, it is nontrivial to analytically solve such a large number of moment equations. Nonetheless, analytical solutions for a quadrupole ionization cooling channel can be worked out in analogy to the standard theory for electron storage rings. In fact, to a large extend, we adapted the electron radiation damping theory for the muon ionization cooling.

This paper is the result of our effort to address longitudinal ionization cooling theory by generalizing our successful treatment of solenoidal transverse ionization cooling channels [6]. Because the dynamics in a solenoidal channel is complicated by the Larmor rotation, we choose quadrupole channels, which are the simplest, for investigating the emittance exchange scheme. This exercise illuminates the beam dynamics of longitudinal ionization cooling. Two new significant heating processes have been found, which are important for longitudinal cooling channel designs.

2. Single-particle dynamics

The dominant forces on the muons are from the electromagnetic field of the focusing channel, i.e., the focusing quadrupoles, bending dipoles, and the longitudinal focusing from RF cavities. This Hamiltonian part of the muon beam dynamics is exactly the same as the quadrupole channels in storage rings and is well described by the Courant–Snyder theory [7]. We consider an idealized uncoupled quadrupole channel with quadrupole strength $K(s)$, horizontal bending radius $\rho(s)$, and RF focusing strength $V(s)$. Using the standard Frenet–Serret coordinates $\{x, y, s\}$, the linear Hamiltonian can be written as

$$H = \frac{1}{2}\left\{P_x^2 + \left[K(s) + \frac{1}{\rho^2}\right]x^2\right\} + \frac{1}{2}\left[P_y^2 - K(s)y^2\right] - \frac{x\delta}{\rho(s)} + \frac{1}{2}\left[\frac{1}{\gamma_0^2}\delta^2 + V(s)z^2\right], \tag{1}$$

where $\{x, P_x\}$, $\{y, P_y\}$ are the horizontal and vertical canonical variables, and $\{z, \delta\}$ are the longitudinal canonical variables representing the relative longitudinal position and momentum deviation $\delta = (P - P_0)/P_0$ from the nominal momentum P_0. We assume a constant nominal momentum, i.e., there is no net acceleration for the reference particle in the cooling channel. γ_0 is the Lorentz factor of the reference particle. The $1/\rho^2$ term arises from the sector-bend focusing.

The primary dissipative and diffusive forces that give the cooling and heating effects are due to the muons' interaction with the material, i.e., inelastic and elastic scattering from the absorber atoms. Both processes are stochastic in nature and cannot be treated with ordinary equations of motion. Similar to radiation damping, the ionization energy loss during inelastic scattering results in an average damping force on a muon that is in opposite direction and proportional to the muon momentum. The elastic scattering does not produce an average force. Both processes yield diffusive (heating) effects known as energy straggling and multiple scattering. All these effects can be treated with stochastic differential equations. The single-particle equations of motion are

$$\frac{\mathrm{d}x}{\mathrm{d}s} = P_x$$

$$\frac{\mathrm{d}P_x}{\mathrm{d}s} = -K_x(s)x + \frac{\delta}{\rho(s)} - \eta(s)P_x + \sqrt{\chi(x,s)}\,\xi_x^{\mathrm{MS}}(s)$$

$$\frac{\mathrm{d}y}{\mathrm{d}s} = P_y$$

$$\frac{\mathrm{d}P_y}{\mathrm{d}s} = -K_y(s)\,y - \eta(s)P_y + \sqrt{\chi(x,s)}\,\xi_y^{\mathrm{MS}}(s)$$

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\delta}{\gamma_0^2} - \frac{x}{\rho(s)}$$
$$\frac{\mathrm{d}\delta}{\mathrm{d}s} = -\mathrm{V}(s)z - (\partial_x\eta)x + \sqrt{\chi_\delta(x,s)}\,\xi_z^{\mathrm{ES}}(s). \tag{2}$$

Here $K_x = K + 1/\rho^2$ and $K_y = -K$ are the magnetic focusing strengths, $\eta(s) = (1/pv)(\mathrm{d}E/\mathrm{d}s)$ is a positive quantity characterizing the cooling force from energy loss, $\chi(x,s) = (13.6\,\mathrm{MeV}/pv)^2(1/L_{\mathrm{rad}})$ is the projected mean-square angular deviation per unit length due to multiple scattering and L_{rad} is the radiation length of absorbers, χ_δ is the mean-square relative energy deviation per unit length due to energy straggling, and $\xi_x^{\mathrm{MS}}(s)$, $\xi_y^{\mathrm{MS}}(s)$ and $\xi_z^{\mathrm{ES}}(s)$ are uncorrelated unit stochastic quantities describing the fluctuating forces due to multiple scattering and energy straggling, respectively. We assume that these stochastic quantities are dominated by Gaussian white noise that satisfy the properties $\langle \xi_i(s) \rangle = 0$ and $\langle \xi_i(s)\xi_j(\bar{s}) \rangle = \delta_{ij}\delta(s-\bar{s})$ for the ensemble average $\langle \cdots \rangle$. The x dependence of the diffusion terms and the $\partial_x\eta$ term is due to the wedged absorber in the horizontal direction for emittance exchange. For simplicity, we treat a uniform density wedged absorber as a uniform thickness absorber with increasing density. χ and χ_δ depend on x through this density variation. Here, we also neglect the weak momentum dependence of ionization energy loss. Without the material terms containing η's and χ's, the equations of motion are a direct result of the Hamiltonian equation (1).

3. Beam-moment equations

Fokker–Planck equations are often derived from stochastic differential equations to solve for average phase-space distribution. When the forces are linear, the Fokker–Plank equation results in closed-form second-order moment equations for the phase-space distribution. Since our interest here is linear cooling dynamics, we derive the moment equations directly from the equations of motion in Eq. (2), by using Eq. (2.42) of Ref. [8] or by straightforward differentiation.

Because the two transverse planes are decoupled and the dynamics of the vertical plane is the same as the dynamics of the horizontal plane if we set the bending radius to infinity and dispersion to zero, it is sufficient to treat only the x–z phase-space dynamics. After some algebra, we obtained the following equations for the 10 second-order beam moments in 4D phase space:

$$\begin{aligned}
\langle x^2 \rangle' &= 2\langle xP_x \rangle \\
\langle xP_x \rangle' &= \langle P_x^2 \rangle - K_x(s)\langle x^2 \rangle + \frac{1}{\rho(s)}\langle x\delta \rangle \\
&\quad - \eta(s)\langle xP_x \rangle \\
\langle P_x^2 \rangle' &= -2K_x(s)\langle xP_x \rangle + \frac{2}{\rho(s)}\langle P_x\delta \rangle \\
&\quad - 2\eta(s)\langle P_x^2 \rangle + \langle \chi(x,s) \rangle \\
\langle z^2 \rangle' &= \frac{2}{\gamma_0^2}\langle z\delta \rangle - \frac{2}{\rho(s)}\langle xz \rangle \\
\langle z\delta \rangle' &= \frac{1}{\gamma_0^2}\langle \delta^2 \rangle - V(s)\langle z^2 \rangle - \frac{1}{\rho(s)}\langle x\delta \rangle \\
&\quad - (\partial_x\eta)\langle xz \rangle \\
\langle \delta^2 \rangle' &= -2V(s)\langle z\delta \rangle - 2(\partial_x\eta)\langle x\delta \rangle \\
&\quad + \langle \chi_\delta(x,s) \rangle \\
\langle xz \rangle' &= \frac{1}{\gamma_0^2}\langle x\delta \rangle + \langle zP_x \rangle - \frac{1}{\rho(s)}\langle x^2 \rangle \\
\langle x\delta \rangle' &= \langle P_x\delta \rangle - V(s)\langle xz \rangle - (\partial_x\eta)\langle x^2 \rangle \\
\langle zP_x \rangle' &= \frac{1}{\gamma_0^2}\langle P_x\delta \rangle - K_x(s)\langle xz \rangle - \frac{1}{\rho(s)}\langle xP_x \rangle \\
&\quad + \frac{1}{\rho(s)}\langle z\delta \rangle - \eta(s)\langle zP_x \rangle \\
\langle P_x\delta \rangle' &= -K_x(s)\langle x\delta \rangle - V(s)\langle zP_x \rangle \\
&\quad + \frac{1}{\rho(s)}\langle \delta^2 \rangle - \eta(s)\langle P_x\delta \rangle \\
&\quad - (\partial_x\eta)\langle xP_x \rangle.
\end{aligned} \tag{3}$$

Hereafter $\langle \cdots \rangle$ denotes phase-space averaging and a prime denotes differentiation with respect to s. Note that, although the stochastic terms depend on x, there is no $\partial_x\chi$ in the moment equations because the stochastic terms in the equations of motion do not have an x component. Furthermore, assuming that the absorber wedges are linear and cover the whole beam,

$$\langle \chi(x,s) \rangle = \chi|_{x=0} + (\partial_x \chi)\langle x \rangle + \frac{1}{2}(\partial_x^2 \chi)\langle x^2 \rangle + \cdots = \chi(s). \tag{4}$$

Thus $\langle \chi(x,s) \rangle$ and $\langle \chi_\delta(x,s) \rangle$ in Eq. (3) can be replaced with their on-axis values $\chi(s)$ and $\chi_\delta(s)$.

The above moment equations are fully coupled and look formidable. However, it is well known that the Hamiltonian part of the transverse and longitudinal dynamics can be decoupled by employing the dispersion function D_x given by the equation [8]

$$D_x'' + K_x(s)D_x = \frac{1}{\rho(s)}. \tag{5}$$

Using the dispersion function and assuming the RF cavities are in dispersion-free regions, the transformation

$$\begin{aligned} x &= \hat{x}_\beta + D_x(s)\delta, \quad P_x = \hat{P}_{x_\beta} + D_x'(s)\delta, \\ z &= \hat{z} - D_x'\hat{x}_\beta + D_x\hat{P}_{x_\beta}, \quad \delta = \hat{\delta} \end{aligned} \tag{6}$$

will decouple the betatron motion $\{\hat{x}_\beta, \hat{P}_{x_\beta}\}$ and the synchrotron motion $\{\hat{z}, \hat{\delta}\}$. In the following we may drop the $\wedge$ or the subscript β to simplify the notation.

Using the moment equation (3), the transformation equation (6), the dispersion equation (5), and the dispersion-free condition that the RF cavities locate at zero-dispersion regions, after some tedious but straightforward algebra, the moment equations in terms of the betatron and synchrotron variables become

$$\begin{aligned}
\langle \hat{x}^2 \rangle' &= 2\langle \hat{x}\hat{P}_x \rangle + 2(\partial_x\eta)D_x \langle \hat{x}^2 \rangle + 2(\partial_x\eta)D_x^2\langle \hat{x}\delta \rangle + D_x^2\chi_\delta(s) \\
\langle \hat{x}\hat{P}_x \rangle' &= \langle \hat{P}_x^2 \rangle - K_1\langle \hat{x}^2 \rangle - \eta_-\langle \hat{x}\hat{P}_x \rangle - \eta_- D_x' \langle \hat{x}\delta \rangle + (\partial_x\eta)D_x^2\langle \hat{P}_x\delta \rangle + D_xD_x'\chi_\delta(s) \\
\langle \hat{P}_x^2 \rangle' &= -2K_1\langle \hat{x}\hat{P}_x \rangle - 2\eta\langle \hat{P}_x^2 \rangle - 2\eta_- D_x'\langle \hat{P}_x\delta \rangle + \chi(s) + (D_x')^2\chi_\delta(s) \\
\langle \hat{z}^2 \rangle' &= 2I_1 \langle \hat{z}\delta \rangle + 2\eta D_x\langle \hat{P}_x\hat{z} \rangle + D_x^2\chi(s) \\
\langle \hat{z}\delta \rangle' &= I_1\langle \delta^2 \rangle - V\langle \hat{z}^2 \rangle - (\partial_x\eta)D_x \langle \hat{z}\delta \rangle - (\partial_x\eta)\langle \hat{x}\hat{z} \rangle + \eta D_x\langle \hat{P}_x\delta \rangle \\
\langle \delta^2 \rangle' &= -2V\langle \hat{z}\delta \rangle - 2(\partial_x\eta)D_x\langle \delta^2 \rangle - 2(\partial_x\eta)\langle \hat{x}\delta \rangle + \chi_\delta(s) \\
\langle \hat{x}\hat{z} \rangle' &= \langle \hat{z}\hat{P}_x \rangle + I_1\langle \hat{x}\delta \rangle + (\partial_x\eta)D_x\langle \hat{x}\hat{z} \rangle + (\partial_x\eta)D_x^2\langle \hat{z}\delta \rangle + \eta D_x\langle \hat{x}\hat{P}_x \rangle \\
\langle \hat{x}\delta \rangle' &= \langle \hat{P}_x\delta \rangle - V\langle \hat{x}\hat{z} \rangle - (\partial_x\eta)\langle \hat{x}^2 \rangle + (\partial_x\eta)D_x^2\langle \delta^2 \rangle - D_x\chi_\delta(s) \\
\langle \hat{z}\hat{P}_x \rangle' &= I_1\langle \hat{P}_x\delta \rangle - K_1\langle \hat{x}\hat{z} \rangle - \eta\langle \hat{z}\hat{P}_x \rangle - \eta_- D_x'\langle \hat{z}\delta \rangle + \eta D_x\langle \hat{P}_x^2 \rangle - D_x\chi(s) \\
\langle \hat{P}_x\delta \rangle' &= -K_1\langle \hat{x}\delta \rangle - V\langle \hat{z}\hat{P}_x \rangle - \eta_+\langle \hat{P}_x\delta \rangle - (\partial_x\eta)\langle \hat{x}\hat{P}_x \rangle - \eta_- D_x'\langle \delta^2 \rangle - D_x'\chi_\delta(s).
\end{aligned} \tag{7}$$

Here $K_1 \equiv K_x - (\partial_x\eta)D_x'$, $I_1 \equiv 1/\gamma_0^2 - D_x/\rho + \eta D_xD_x'$, and $\eta_\pm \equiv \eta \pm (\partial_x\eta)D_x$. Despite the appearance, these equations are simpler than Eq. (3) since the transverse and longitudinal parts are decoupled if we neglect the material terms containing η's and χ's. These terms are small and can be treated as perturbation, as has been shown in solenoidal transverse cooling channels [6].

4. Beam-envelope equations

We now introduce beam emittances and envelope functions, which characterize the density and shape of beam phase-space distribution. For a matched beam, the envelope functions are determined by the lattice functions characterizing the focusing channel. Similar to the electron ring theory, we introduce the transverse and longitudinal envelope functions as

$$\begin{bmatrix} \langle x_\beta^2 \rangle & \langle x_\beta P_{x_\beta} \rangle \\ \langle x_\beta P_{x_\beta} \rangle & \langle P_{x_\beta}^2 \rangle \end{bmatrix} \equiv \varepsilon_x \begin{bmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{bmatrix} \tag{8}$$

and

$$\begin{bmatrix} \langle \hat{z}^2 \rangle & \langle \hat{z}\delta \rangle \\ \langle \hat{z}\delta \rangle & \langle \delta^2 \rangle \end{bmatrix} \equiv \varepsilon_z \begin{bmatrix} \beta_z & -\alpha_z \\ -\alpha_z & \gamma_z \end{bmatrix}. \tag{9}$$

We also introduce a new emittance ε_c for the cross-space between the transverse and longitudinal phase space as

$$\begin{bmatrix} \langle x_\beta \hat{z} \rangle & \langle x_\beta \delta \rangle \\ \langle \hat{z} P_{x_\beta} \rangle & \langle P_{x_\beta} \delta \rangle \end{bmatrix} \equiv \varepsilon_c \left(\begin{bmatrix} \beta_c & -\alpha_c \\ -\alpha_c & \gamma_c \end{bmatrix} \cos\phi + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sin\phi \right). \quad (10)$$

In all three cases, we require that the emittance square is equal to the determinant of the corresponding moment blocks and $\beta\gamma - \alpha^2 = 1$. Inserting these envelope functions into the moment equations, one could obtain a set of differential equations for the emittances and envelope functions that is equivalent to the set of moment equations. However, the exact envelope equations are too complicated and not very useful. Instead, we will focus on a much simpler but sufficiently good approximation similar to the analysis of the transverse solenoidal cooling channel. The key is that the muons' interaction with material is weak and only the first-order material terms need to be retained [6].

As a zero-order approximation, we consider the case without dissipative and diffusive forces from the material. Dropping the material terms from the moment equations, it can be shown that

$$\begin{aligned}
&\varepsilon_x' = \varepsilon_z' = \varepsilon_c' = 0 \\
&\beta_x' = -2\,\alpha_x \\
&\alpha_x' = K_x\beta_x - \gamma_x \\
&\beta_z' = -2I\alpha_z \\
&\alpha_z' = V\beta_z - I\gamma_z \\
&\beta_c' = -(1+I)\,\alpha_c + (\beta_c\phi' - 1 + I)\tan\phi \\
&\alpha_c' = \tfrac{1}{2}(K_x + V)\,\beta_c - \tfrac{1}{2}(1+I)\gamma_c + \alpha_c \tan\phi\,\phi' \\
&\phi' = \tfrac{1}{2}\left[(K_x - V)\,\beta_c + (1-I)\,\gamma_c\right].
\end{aligned} \quad (11)$$

Here $I = 1/\gamma_0^2 - D_x/\rho$ is negative of the usual slip factor. All three emittances are conserved as expected. However, unlike ε_x and ε_z that can have any values for a matched beam $\varepsilon_c = 0$. This is important and is the main motivation to introduce ε_c. The envelope functions β_c, α_c, and ϕ are not of much interest here since we assume a matched beam.

Even considering the material effects, the lattice functions determined by Eq. (11) still provide a good approximation of the beam-envelope functions of the phase-space distribution. However, the beam emittances will be changed considerably by the cooling in the material. To examine the cooling process, we study the first-order perturbation of material effects on the emittances. The calculation of the emittance change rate can be considerably simplified by a simple but important observation:

$$\varepsilon' = \sum \{\text{material term containing } \eta \text{ or } \chi\} \times \{\varepsilon\} \times \{\beta, \alpha, \text{etc. if any}\}. \quad (12)$$

Thus, to the first-order material effects, only zero-order emittances and envelope functions are required for calculating the emittance change rate. Therefore, it is sufficient to use Eq. (11) to determine the envelope functions. Using the moment equations in Eq. (7) to compute $\gamma_x\langle \hat{x}^2\rangle' + 2\alpha_x\langle \hat{x}\hat{P}_x\rangle' + \beta_x\langle \hat{P}_x^2\rangle'$ and $\gamma_z\langle \hat{z}^2\rangle' + 2\alpha_z\langle \hat{z}\delta\rangle' + \beta_z\langle \delta^2\rangle'$ yield ε_x' and ε_z' as

$$\varepsilon_x' = -[\eta - (\partial_x\eta)D_x]\varepsilon_x + \tfrac{1}{2}\beta_x\chi + \tfrac{1}{2}\mathscr{H}_x\chi_\delta \quad (13)$$

and

$$\varepsilon_z' = -[\partial_\delta\eta + (\partial_x\eta)D_x]\varepsilon_z + \tfrac{1}{2}\beta_z\chi_\delta + \tfrac{1}{2}\gamma_z D_x^2\chi \quad (14)$$

where $\mathscr{H}_x \equiv \gamma_x D_x^2 + 2\alpha_x D_x D_x' + \beta_x D_x'^2$. Here, for completeness, we added the usually weak term $\partial_\delta\eta$ due to the momentum dependence of ionization energy loss. This term is obvious and has been consistently treated in moment equations although it has not been presented in our previous equations. The change rate for ε_c is complicated but it has the form $\varepsilon_c' = \{\cdots\}\sin\phi + \{\cdots\}\cos\phi$. Thus averaging over ϕ leads to $\varepsilon_c' \simeq 0$.

5. Emittance evolution and equilibrium emittances

Eqs. (13) and (14) are the key equations governing the emittance evolution in the ionization cooling process. They are simple and not coupled at all. In case the dispersion is zero, these equations reduce to the well-known equations for a straight transverse quadrupole cooling channel [5]. The emittance exchange is achieved by the term $(\partial_x\eta)D_x$. It increases the longitudinal cooling rate and reduces the transverse cooling rate by the same amount, a reflection of Robinson's theorem of radiation damping. The two indispensable ingredients for emittance exchange, dispersion D_x and wedged absorber represented by $\partial_x\eta$, show up here in a simple product. The third term in the transverse equation is the well-known heating term due to multiple scattering. The $\beta_z\chi_\delta$ term to the longitudinal emittance is the $\beta_x\chi$ term to the transverse emittance. The last terms in both equations have not been addressed in the literature. Both are extra heating terms and need to be carefully controlled in cooling channel design. The $\mathcal{H}_x$ is a familiar term in radiation damping theory. It characterizes the heating due to energy straggling. The $\gamma_z D_x^2$ term is similar to the multiple scattering term except that the beam size is due to the spatial spread from the energy fluctuation (note that $\gamma_z D_x^2\varepsilon_z = D_x^2\sigma_\delta^2$).

Although the coefficients of the four heating terms look different, they are easy to understand from a common feature: each term is a noise contribution to the invariant $\langle \gamma q^2 + 2\alpha qp + \beta p^2 \rangle$ of the phase space (q, p). For example, energy straggling results in energy fluctuation $\sigma_\delta^2 \propto \chi_\delta$, which leads to fluctuations in both position and momentum of the transverse phase space through the dispersion D_x and D_x'. Thus it contributes the term $\mathcal{H}_x\chi_\delta$ to the transverse invariant. Therefore, the $\mathcal{H}_x$ is just a reflection of the invariant structure. Similarly, the multiple scattering results in a momentum fluctuation $\sigma_{P_x}^2 \propto \chi$, which leads to longitudinal position (but no energy) fluctuation through D_x. Thus it contributes the term $\gamma_z D_x^2\chi$ to the longitudinal invariant.

Since the two emittance equations are not coupled and both are first-order inhomogeneous differential equations, they can be integrated to a closed form

$$\varepsilon(s) = \varepsilon(0)e^{-\int_0^s \Lambda(\bar{s})\,d\bar{s}} + e^{-\int_0^s \Lambda(\bar{s})\,d\bar{s}} \times \int_0^s d\bar{s}\, e^{\int_0^{\bar{s}} \Lambda(\tilde{s})\,d\tilde{s}}\,\Xi(\bar{s}). \tag{15}$$

For transverse emittance, the cooling and heating rates are

$$\Lambda(s) = \eta - (\partial_x\eta)D_x, \qquad \Xi(s) = \tfrac{1}{2}\beta_x\chi + \tfrac{1}{2}\mathcal{H}_x\chi_\delta. \tag{16}$$

For longitudinal emittance

$$\Lambda(s) = \partial_\delta\eta + (\partial_x\eta)D_x,$$

$$\Xi(s) = \tfrac{1}{2}\beta_z\chi_\delta + \tfrac{1}{2}\gamma_z D_x^2\chi. \tag{17}$$

All these cooling and heating rates can be calculated with given lattice functions and absorber properties, thus the emittance evolution of a matched beam in a quadrupole channel can be computed using the closed-form solutions. For a sufficiently long cooling channel, the beam emittances approach certain equilibrium values given by the second term of Eq. (15).

In case of periodic cooling channels, the emittance evolution can be characterized by integrals over one period. Let λ be the period length, then Eq. (15) can be written as

$$\begin{aligned}\varepsilon(m\lambda) &= e^{-m\Gamma(\lambda)}\varepsilon(0) + \{e^{-(m-1)\Gamma(\lambda)} + e^{-(m-2)\Gamma(\lambda)} + \cdots + 1\}\mathcal{W}(\lambda)\\ &= e^{-m\Gamma(\lambda)}\varepsilon(0) + \frac{1-e^{-m\Gamma(\lambda)}}{1-e^{-\Gamma(\lambda)}}\mathcal{W}(\lambda),\end{aligned} \tag{18}$$

where

$$\Gamma(s) = \int_0^s \Lambda(\bar{s})\,d\bar{s} \quad \text{and}$$

$$\mathcal{W}(s) = e^{-\Gamma(s)}\int_0^s e^{\Gamma(\bar{s})}\,\Xi(\bar{s})\,d\bar{s}. \tag{19}$$

From Eq. (18), the cooling process is clear. The initial emittance $\varepsilon(0)$ is exponentially damped while extra emittance $\mathcal{W}(\lambda)$ is generated in each period and then damped by a factor $e^{-\Gamma(\lambda)}$ in each of the succeeding periods. As $m \to \infty$, assuming a positive damping rate, an equilibrium will be reached with equilibrium emittance given by

$$\varepsilon(\infty) = \frac{\mathcal{W}(\lambda)}{1-e^{-\Gamma(\lambda)}} \simeq \frac{\mathcal{W}(\lambda)}{\Gamma(\lambda)}. \tag{20}$$

Similar to synchrotron radiation integrals, we introduce a set of integrals to specify the ionization cooling process:

$$\zeta_1 = \int_0^\lambda ds\,\eta(s) \tag{21}$$

$$\zeta_3 = \int_0^\lambda ds\,\partial_x\eta D_x \tag{22}$$

$$\zeta_4 = \int_0^\lambda \mathrm{d}s\,\partial_\delta \eta \tag{23}$$

$$\begin{aligned}\mathscr{W}_1 &= \frac{1}{2}\mathrm{e}^{-\int_0^\lambda [\eta-(\partial_x\eta)D_x]\,\mathrm{d}\bar{s}} \int_0^\lambda \mathrm{d}s\,\mathrm{e}^{\int_0^s [\eta-(\partial_x\eta)D_x]\,\mathrm{d}\bar{s}} \beta_x\chi \\ &\simeq \frac{1}{2}\int_0^\lambda \mathrm{d}s \beta_x\chi \end{aligned}\tag{24}$$

$$\begin{aligned}\mathscr{W}_2 &= \frac{1}{2}\mathrm{e}^{-\int_0^\lambda [\partial_\delta\eta+(\partial_x\eta)D_x]\,\mathrm{d}\bar{s}} \\ &\quad\times \int_0^\lambda \mathrm{d}s\,\mathrm{e}^{\int_0^s [\partial_\delta\eta+(\partial_x\eta)D_x]\,\mathrm{d}\bar{s}} \beta_z\chi_\delta \\ &\simeq \frac{1}{2}\int_0^\lambda \mathrm{d}s\,\beta_z\chi_\delta \end{aligned}\tag{25}$$

$$\begin{aligned}\mathscr{W}_3 &= \frac{1}{2}\mathrm{e}^{-\int_0^\lambda [\eta-(\partial_x\eta)D_x]\,\mathrm{d}\bar{s}} \int_0^\lambda \mathrm{d}s\,\mathrm{e}^{\int_0^s [\eta-(\partial_x\eta)D_x]\,\mathrm{d}\bar{s}} \mathscr{H}_x\chi_\delta \\ &\simeq \frac{1}{2}\int_0^\lambda \mathrm{d}s \mathscr{H}_x\chi_\delta \end{aligned}\tag{26}$$

$$\begin{aligned}\mathscr{W}_4 &= \frac{1}{2}\mathrm{e}^{-\int_0^\lambda [\partial_\delta\eta+(\partial_x\eta)D_x]\,\mathrm{d}\bar{s}} \\ &\quad \int_0^\lambda \mathrm{d}s\,\mathrm{e}^{\int_0^s [\partial_\delta\eta+(\partial_x\eta)D_x]\,\mathrm{d}\bar{s}} \gamma_z D_x^2\chi \\ &\simeq \frac{1}{2}\int_0^\lambda \mathrm{d}s\,\gamma_z D_x^2\chi. \end{aligned}\tag{27}$$

To avoid confusion with radiation integrals and be consistent with our notations for solenoidal cooling channels, we used ζ and $\mathscr{W}$ instead of I for these integrals. The ionization integrals ζ's characterize the cooling rates. ζ_1 and ζ_2 have been used for solenoidal channels [6]. $\mathscr{W}$'s give the emittance generated in one period by the four different heating mechanisms. Multiple-scattering integrals $\mathscr{W}_1$ and $\mathscr{W}_4$ give multiple-scattering heating to the transverse and longitudinal emittances, respectively. Similarly $\mathscr{W}_2$ and $\mathscr{W}_3$ may be referred to as energy-straggling integrals.

In terms of these integrals, the equilibrium emittances can be expressed as

$$\varepsilon_x^\infty \simeq \frac{\mathscr{W}_1+\mathscr{W}_3}{\zeta_1-\zeta_3} \quad \text{and} \quad \varepsilon_z^\infty \simeq \frac{\mathscr{W}_2+\mathscr{W}_4}{\zeta_3+\zeta_4}. \tag{28}$$

Evolution towards this equilibrium is described by Eq. (18). The damping times τ_x and τ_z simply read

$$\frac{1}{\tau_x} = \frac{v_u}{\lambda}(\zeta_1-\zeta_3) \quad \text{and} \quad \frac{1}{\tau_z} = \frac{v_u}{\lambda}(\zeta_3+\zeta_4) \tag{29}$$

where v_u is the muon's velocity. The transverse and longitudinal cooling lengths $\tau_{x,z}v_u$ are $\lambda(\zeta_1-\zeta_3)$ and $\lambda(\zeta_3+\zeta_4)$.

To get a sense of the importance of the new heating terms $\mathscr{W}_3$ and $\mathscr{W}_4$, let us compare them with the more familiar terms $\mathscr{W}_1$ and $\mathscr{W}_2$. Roughly speaking, the parameters considered in current cooling channel designs are around $\chi/\chi_\delta \simeq 5$–10, $\beta_x \simeq 0.25$ m, $D_x \simeq 0.5$ m, $\mathscr{H}_x \simeq 1$, $\beta_z \sim 0.5$ m, and $\gamma_z \sim 2$. Thus $\mathscr{W}_3$ will probably be comparable but smaller than the well-known transverse heating term $\mathscr{W}_1$. However, longitudinal heating due to multiple scattering, $\mathscr{W}_4$, can be larger than the energy-straggling term $\mathscr{W}_2$, especially if longitudinal focusing is strong. Since the two new heating terms depend quadratically on the dispersion, they could get much worse if the dispersion becomes too large. On the other hand, efficient longitudinal cooling requires a large dispersion. Therefore, dispersion needs to be carefully optimized for longitudinal cooling.

6. Conclusion and discussions

We have developed a linear theory for transverse and longitudinal ionization cooling of a matched beam in a quadrupole channel. Simple emittance evolution equations are obtained. Various cooling and heating effects are systematically deduced and new heating terms are identified. Multiple-scattering and energy-straggling integrals are introduced to characterize these effects. Although quadrupole channels may not be the best choice for muon cooling, this simple theory illustrates many common features of ionization cooling and should be useful for theoretical understanding of ionization cooling dynamics.

A general theory on beam equilibrium emittances in electron rings was derived by Ruggiero et al. [9] based on the linearized Fokker–Planck equation. Applied to our ionization cooling problem, the dissipation matrix A and noise matrix B in that paper are, respectively,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \partial_x \eta & 0 & 0 & \partial_\delta \eta \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi_\delta \end{bmatrix}. \tag{30}$$

We then confirmed the equilibrium emittances in Eq. (28).

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