Lecture 9

Microwave Network Analysis

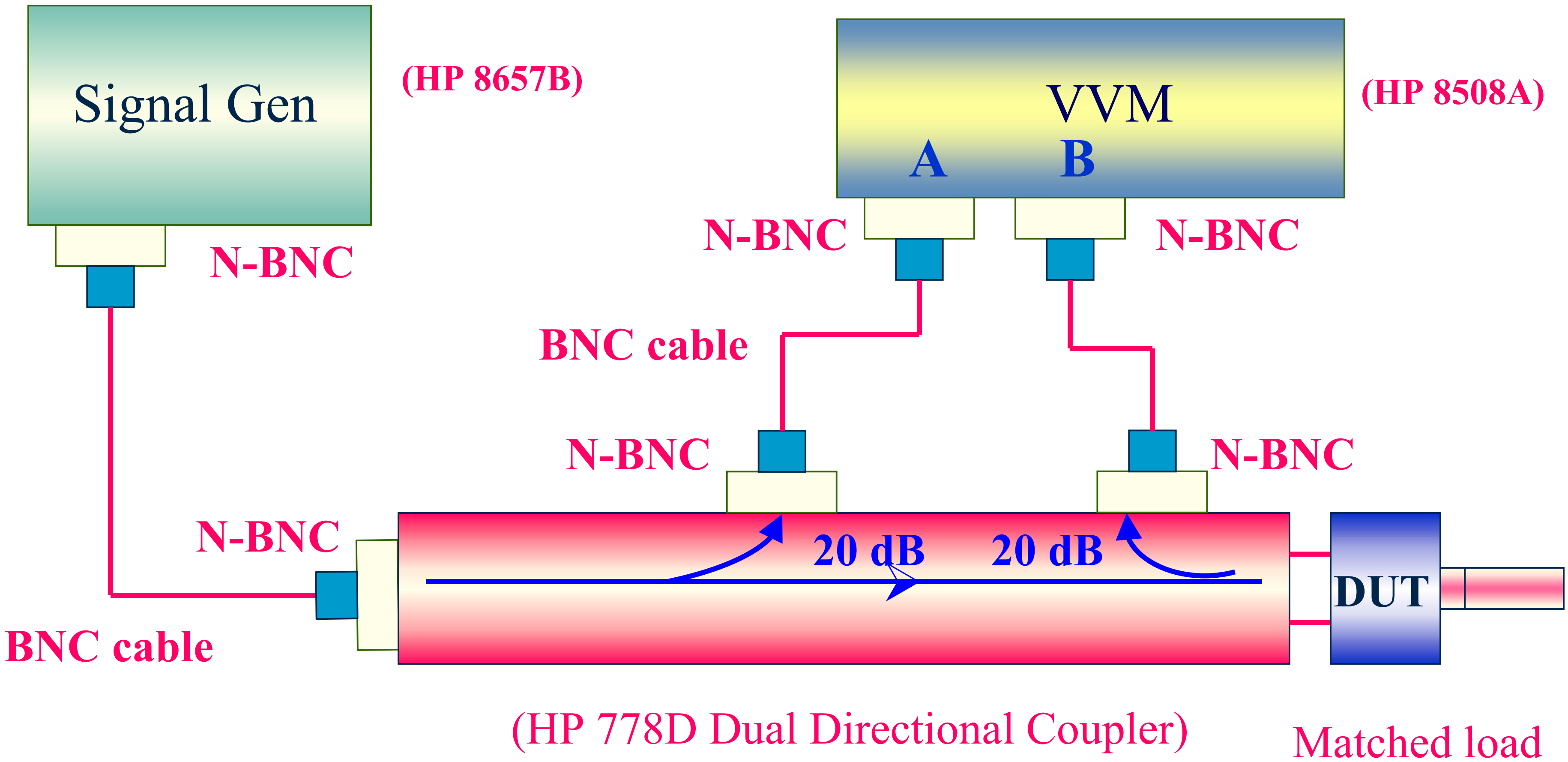
December 3, 2002

S-Parameter Measurement Technique

VVM: The vector voltmeter measures the magnitude of a reference and test voltage and the difference in phase between the voltages. Because it can measure phase, it allows us to directly measure the S-parameters of a circuit

Unfortunately, the use of the directional couplers and test cables connecting the measuring system to the vector voltmeter introduces unknown attenuation and phase shift into the measurements. These can be compensated for by making additional "calibration" measurements.

Reflection measurements: S_{11} or S_{22}



Reflection measurements: S_{11} or S_{22}

From the setup, it is seen that the voltage at channel A of the VVM (A^D) is proportional to the amplitude of the voltage wave entering the device under test (DUT) (a^D_1). Similarly, the voltage at channel B (B^D) is proportional to the amplitude of the voltage wave reflected from DUT (b^D_1). Thus we can write

$$A^D = K_A a_1^D$$

$$B^D = K_B b_1^D$$

Where K_A and K_B are constants that depend on the connecting cables. Since a^D_2 is zero because of the matched load at port 2, S11 is given by

$$S_{11} = \frac{b_1^D}{a_1^D} = \frac{B^D / K_B}{A^D / K_A}$$

Reflection measurements: S_{11} or S_{22}

To find K_A and K_B it is necessary to make a second measurement with a known DUT. This is called a “calibration” measurement. If the DUT is removed and replaced by a short circuit, the voltage at channel A (A^s) and channel B(B^s) are given by

$$A^S = K_A a_1^S$$

$$B^S = K_B b_1^S$$

Where a^s_1 is the amplitude of the voltage wave entering the short and b^s_1 is the amplitude of the voltage wave reflected from the short. However, for a short circuit the ratio of these amplitudes is –1 (reflection coefficient of a short). Thus

$$\frac{b_1^S}{a_1^S} = \frac{B^S / K_B}{A^S / K_A} = -1$$

Reflection measurements: S_{11} or S_{22}

$$\frac{K_B}{K_A} = -\frac{B^S}{A^S}$$

$$S_{11} = -\frac{\left(\frac{B^D}{A^D}\right)}{\left(\frac{B^S}{A^S}\right)}$$

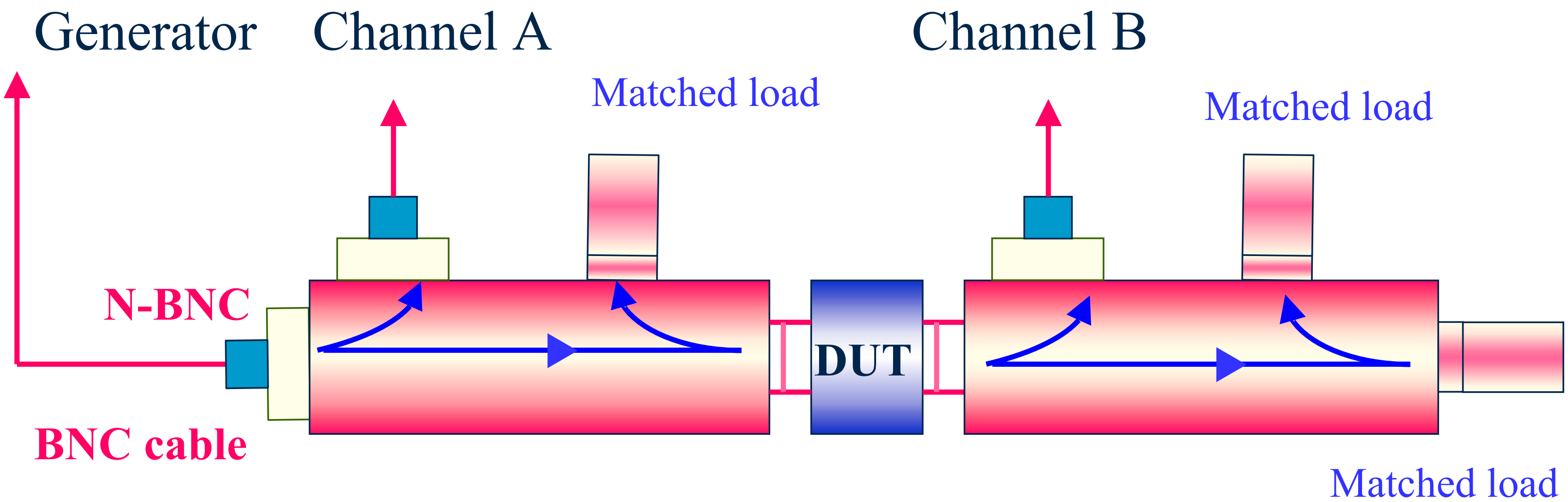
Note: since VVM displays quantities in terms of magnitude and phase we can rewrite S_{11} as

$$\left(\frac{B^D}{A^D}\right) = \Gamma^D \angle \phi^D$$

$$S_{11} = \frac{\Gamma^D}{\Gamma^S} \angle\left(\phi^D - \phi^S - \pi\right)$$

$$\left(\frac{B^S}{A^S}\right) = \Gamma^S \angle \phi^S$$

Transmission measurements: S_{12} or S_{21}



The DUT is connected directly between two directional couplers. Voltage at A of the VVM is proportional to the voltage wave incident on the DUT while the voltage at B of the VVM is proportional to voltage wave transmitted through the DUT.

Transmission measurements: S_{12} or S_{21}

$$A^D = K_A a_1^D$$
$$B^D = K_B b_2^D \quad \Rightarrow \quad S_{21} = \frac{b_2^D}{a_1^D} = \frac{B^D / K_B}{A^D / K_A}$$

To find out the constants a calibration measurement must be made. Remove the DUT and connect both directional couplers directly together. The Known DUT in this case is just a zero-length guide with a transmission coefficient of unity. The measured voltages are:

$$A^E = K_A a_1^E$$
$$B^E = K_B b_2^E$$

where

$$\frac{b_2^E}{a_1^E} = \frac{B^E / K_B}{A^E / K_A} = 1$$

$$\therefore \frac{K_B}{K_A} = \frac{B^E}{A^E}$$

Transmission measurements: S_{12} or S_{21}

$$S_{21} = -\frac{\left(\frac{B^D}{A^D}\right)}{\left(\frac{B^E}{A^E}\right)}$$

$$S_{21} = \frac{T^D}{T^E} \angle\left(\theta^D - \theta E\right)$$

where

$$\left(\frac{B^D}{A^D}\right) = T^D \angle \theta^D$$

$$\left(\frac{B^E}{A^E}\right) = T^E \angle \theta^E$$

Scattering Parameters

- **Scattering Parameters (S-Parameters) plays a major role is network analysis**

- **This importance is derived from the fact that practical system characterizations can no longer be accomplished through simple open- or short-circuit measurements, as is customarily in low-frequency applications.**

- **In the case of a short circuit with a wire; the wire itself possesses an inductance that can be of substantial magnitude at high frequency.**

- **Also open circuit leads to capacitive loading at the terminal.**

Scattering Parameters

- **In either case, the open/short-circuit conditions needed to determine Z-, Y-, h-, and ABCD-parameters can no longer be guaranteed.**

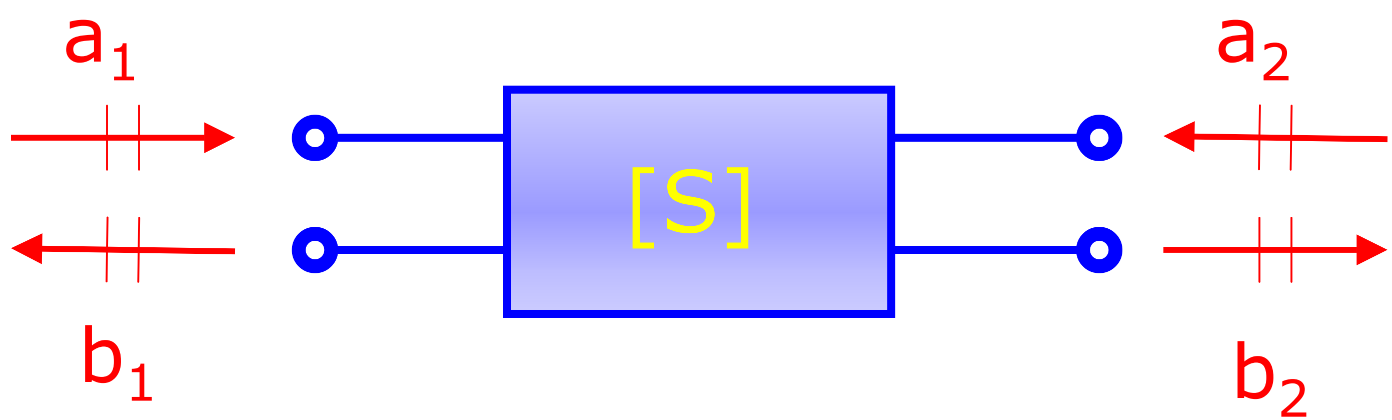
- **Moreover, when dealing with wave propagation phenomena, it is not desirable to introduce a reflection coefficient whose magnitude is unity.**

- **For instance, the terminal discontinuity will cause undesirable voltage and/or current wave reflections, leading to oscillation that can result in the destruction of the device.**

- **With S-parameters, one has proper tool to characterize the two-port network description of practically all RF devices without harm to DUT.**

Definition of Scattering Parameters

- **S-parameters are power wave descriptors that permit us to define the input-output relations of a network in terms of incident and reflected power waves.**



a_n – **normalized incident power waves**

b_n – **normalized reflected power waves**

Definition of Scattering Parameters

$$a_n = \frac{1}{2\sqrt{Z_\circ}}(V_n + Z_\circ I_n) \qquad (1)$$

$$b_n = \frac{1}{2\sqrt{Z_\circ}}(V_n - Z_\circ I_n) \qquad (2)$$

- **Index n refers either to port number 1 or 2. The impedance Z_0 is the characteristic impedance of the connecting lines on the input and output side of the network.**

Definition of Scattering Parameters

- **Inverting (1) leads to the following voltage and current expressions:**

$$V_n = \sqrt{Z_\circ}\left(a_n + b_n\right) \quad (3)$$

$$I_n = \frac{1}{\sqrt{Z_\circ}}\left(a_n - b_n\right) \quad (4)$$

Definition of Scattering Parameters

- **Recall the equations for power:**

$$P_n = \frac{1}{2} Re\left\{V_n I_n^*\right\} = \frac{1}{2}\left(\left|a_n\right|^2 - \left|b_n\right|^2\right) \quad (5)$$

- **Isolating forward and backward traveling wave components in (3) and (4), we see**

$$a_n = \frac{V_n^+}{\sqrt{Z_\circ}} = \sqrt{Z_\circ} I_n^+ \qquad (6)$$

$$b_n = \frac{V_n^-}{\sqrt{Z_\circ}} = -\sqrt{Z_\circ} I_n^- \quad (7)$$

Definition of Scattering Parameters

- **We can now define S-parameters:**

$$\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \quad (8)$$

Definition of Scattering Parameters

$$S_{11}=\left.\frac{b_1}{a_1}\right|_{a_2=0}=\frac{\text{Refkected powe wave at port 1}}{\text{Incident power wave at port 2}} \tag{9}$$

$$S_{21}=\left.\frac{b_2}{a_1}\right|_{a_2=0}=\frac{\text{Transmitted powe wave at port 2}}{\text{Incident power wave at port 1}} \tag{10}$$

$$S_{22}=\left.\frac{b_2}{a_2}\right|_{a_1=0}=\frac{\text{Refkected powe wave at port 2}}{\text{Incident power wave at port 2}} \tag{11}$$

$$S_{12}=\left.\frac{b_1}{a_2}\right|_{a_1=0}=\frac{\text{Transmitted powe wave at port 1}}{\text{Incident power wave at port 2}} \tag{12}$$

➤ Observations:

⌗ $a_2=0$, and $a_1=0 \Rightarrow$ no power waves are returned to the network at either port 2 or port 1.

⌗ However, these conditions can only be ensured when the connecting transmission line are terminated into their characteristic impedances.

⌗ Since the S-parameters are closely related to power relations, we can express the normalized input and output waves in terms of time averaged power.

⌗ The average power at port 1 is given by

$$P_1 = \frac{1}{2}\frac{\left|V_1^+\right|^2}{Z_\circ}\left(1-\left|\Gamma_{in}\right|^2\right) = \frac{1}{2}\frac{\left|V_1^+\right|^2}{Z_\circ}\left(1-\left|S_{11}\right|^2\right) \quad (13)$$

Scattering Parameters

- **The reflection coefficient at the input side is expressed in terms of S_{11} under matched output according:**

$$\Gamma_{in} = \frac{V_1^-}{V_1^+} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = S_{11} \qquad (14)$$

- **This also allow us to redefine the VSWR at port 1 in terms of S_{11} as**

$$VSWR = \frac{1+|S_{11}|}{1-|S_{11}|} \qquad (15)$$

Scattering Parameters

- **We can identify the incident power in (13) and express it in terms of a_1:**

$$\frac{1}{2}\frac{\left|V_1^+\right|^2}{Z_\circ} = P_{inc} = \frac{\left|a_1\right|^2}{2} \qquad (16)$$

Maximal available power from the generator

- **The total power at port 1 (under matched output condition) expressed as a combination of incident and reflected powers:**

$$P_1 = P_{inc} + P_{refl} = \frac{1}{2}\left(\left|a_1\right|^2 - \left|b_1\right|^2\right) = \frac{\left|a_1\right|^2}{2}\left(1 - \left|\Gamma_{in}\right|^2\right) \qquad (17)$$

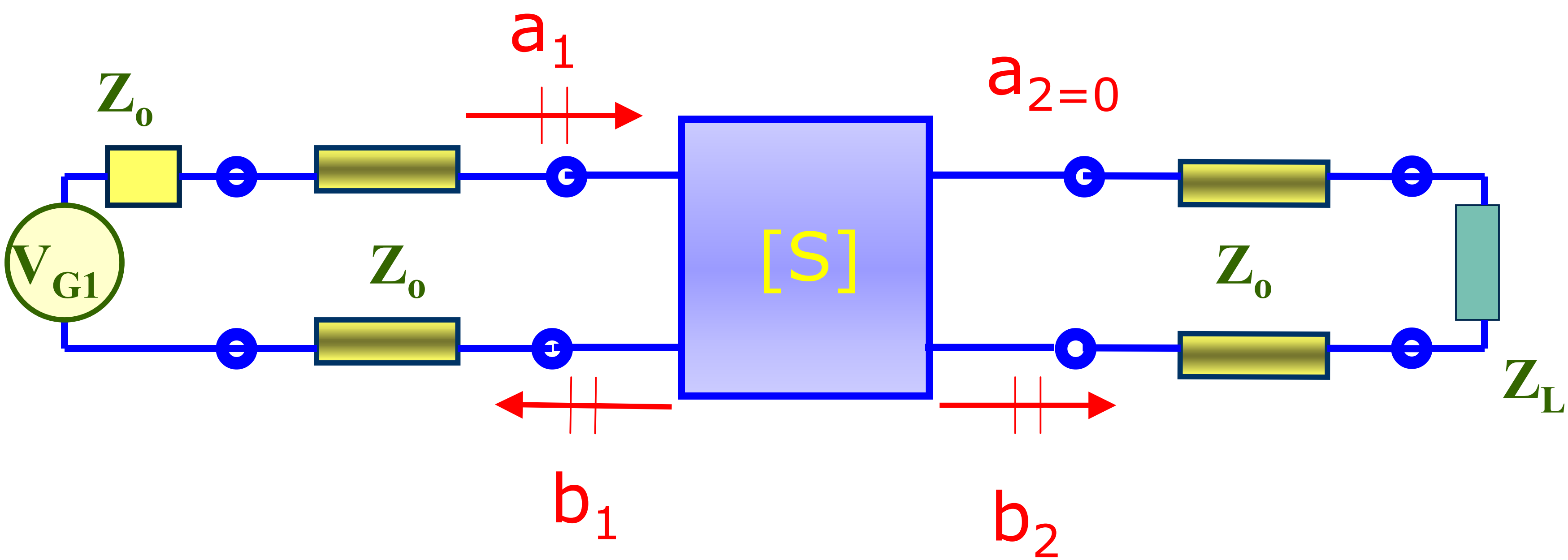
Scattering Parameters

- **If the reflected coefficient, or S_{11}, is zero, all available power from the source is delivered to port 1 of the network. An identical analysis at port 2 gives**

$$P_2 = \frac{1}{2}\left(\left|a_2\right|^2 - \left|b_2\right|^2\right) = \frac{\left|a_2\right|^2}{2}\left(1 - \left|\Gamma_{out}\right|^2\right) \qquad (18)$$

Meaning of S-Parameters

- **S-parameters can only be determined under conditions of perfect matching on the input or the output side.**



Measurement of S_{11} and S_{21} by matching the line impedance Z_o at port 2 through a corresponding load impedance $Z_L = Z_o$

Meaning of S-Parameters

- **This configuration allows us to compute S_{11} by finding the input reflection coefficient:**

$$S_{11} = \Gamma_{in} = \frac{Z_{in} - Z_{\circ}}{Z_{in} + Z_{\circ}} \qquad (19)$$

- **Taking the logarithm of the magnitude of S_{11} gives us the return loss in dB**

$$RL = -20\, log\left|S_{11}\right| \qquad (20)$$

Meaning of S-Parameters

- **With port 2 properly terminated, we find**

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = \left.\frac{V_2^- \big/ \sqrt{Z_\circ}}{\left(V_1 + Z_\circ I_1\right)\big/\left(2\sqrt{Z_\circ}\right)}\right|_{I_2^+ = V_2^+ = 0} \qquad (21)$$

- **Since a_2=0, we can set to zero the positive traveling voltage and current waves at port 2.**

- **Replacing V_1 by the generator voltage V_{G1} minus the voltage drop over the source impedance Z_o, V_{G1}-Z_oI_1 gives**

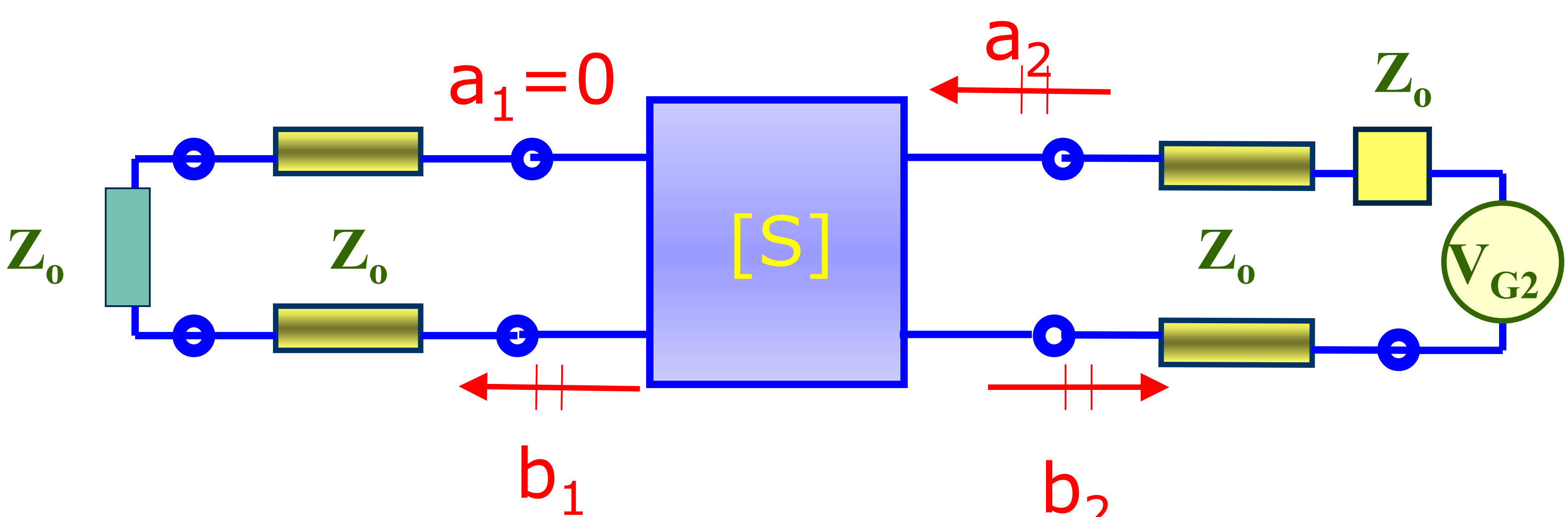
$$S_{21} = \frac{2V_2^-}{V_{G1}} = \frac{2V_2}{V_{G1}} \qquad (22)$$

Meaning of S-Parameters

- **The forward power gain is**

$$G_\circ = |S_{21}|^2 = \left|\frac{V_2}{V_{G1}/2}\right|^2 \qquad (23)$$

- **If we reverse the measurement procedure and attach a generator voltage V_{G2} to port 2 and properly terminate port 1, we can determine the remaining two S-parameters, S_{22} and S_{12}.**



Meaning of S-Parameters

- **To compute S_{22} we need to find the output reflection coefficient Γ_{out} in a similar way for S_{11}:**

$$S_{22} = \Gamma_{out} = \frac{Z_{out} - Z_\circ}{Z_{out} + Z_\circ} \qquad (24)$$

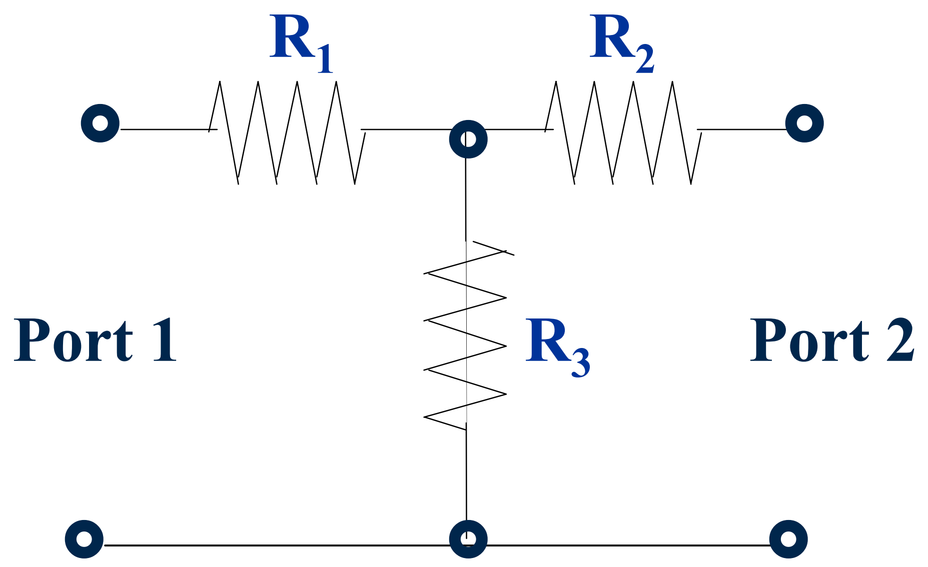
$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} = \left.\frac{V_1^- / \sqrt{Z_\circ}}{(V_2 + Z_\circ I_2)/(2\sqrt{Z_\circ})}\right|_{I_1^+=V_1^+=0} \qquad (25)$$

$$S_{12} = \frac{2V_1^-}{V_{G2}} = \frac{2V_1}{V_{G2}} \qquad (26) \qquad\qquad G_{or} = |S_{12}|^2 = \left|\frac{V_1}{V_{G2}/2}\right|^2 \qquad (27)$$

Reverse power gain

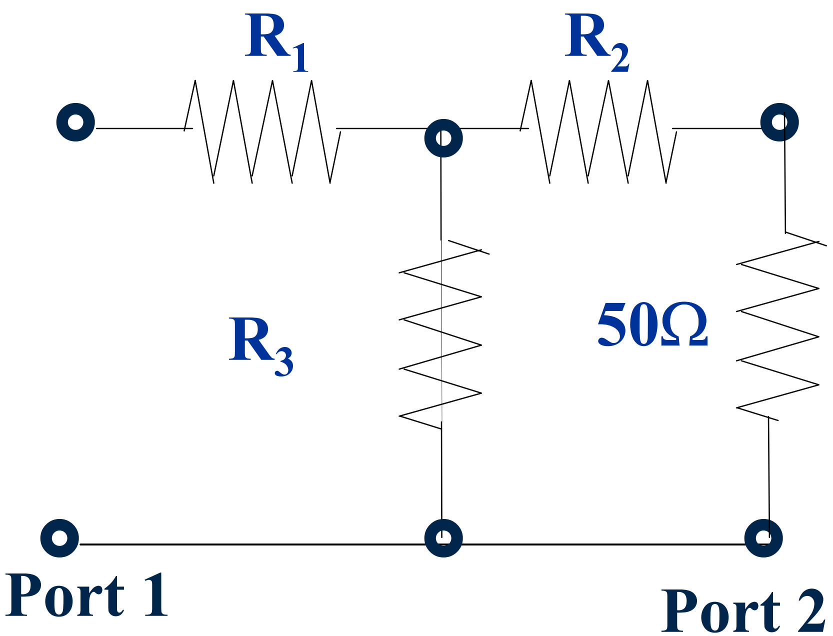
Determination of a T-network elements

Find the S-parameters and resistive elements for the 3-dB attenuator network. Assume that the network is placed into a transmission line section with a characteristic line impedance of Z_o=50 Ω



Determination of a T-network elements

An attenuator should be matched to the line impedance and must meet the requirement $S_{11} = S_{22} = 0$.



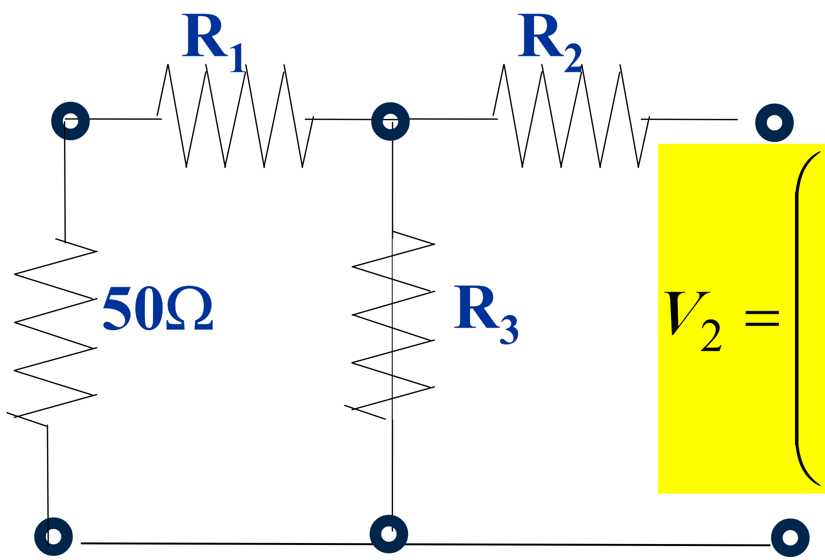
Circuit for S_{11} and S_{21}

$$Z_{in} = R_1 + \frac{R_3(R_2 + 50\Omega)}{(R_3 + R_2 + 50\Omega)} = 50\Omega$$

Because of symmetry, it is clear that $R_1 = R_2$.

Determination of a T-network elements

We now investigate the voltage $V_2 = V^-_2$ at port 2 in terms of $V_1=V^+_1$.



$$V_2 = \left(\frac{\dfrac{R_3\left(R_1 + 50\Omega\right)}{\left(R_3 + R_1 + 50\Omega\right)}}{\dfrac{R_3\left(R_1 + 50\Omega\right)}{\left(R_3 + R_1 + 50\Omega\right)} + R_1} \right) \left(\frac{50\Omega}{50\Omega + R_1} \right) V_1$$

Port 1 **Port 2**

Determination of a T-network elements

For a 3 dB attenuation, we require

$$S_{21} = \frac{2V_2}{V_{G1}} = \frac{V_2}{V_1} = \frac{1}{\sqrt{2}} = 0.707 = S_{12}$$

Setting the ratio of V_2/V_1 to 0.707 and using the input impedance expression, we can determine R_1 and R_3

$$R_1 = R_2 = \frac{\sqrt{2}-1}{\sqrt{2}+1} Z_\circ = 8.58\Omega$$

$$R_3 = 2\sqrt{2} Z_\circ = 141.4\Omega$$

Determination of a T-network elements

Note: *the choice of the resistor network ensures that at the input and output ports an impedance of 50 Ω is maintained. This implies that this network can be inserted into a 50 Ω transmission line section without causing undesired reflections, resulting in an insertion loss.*

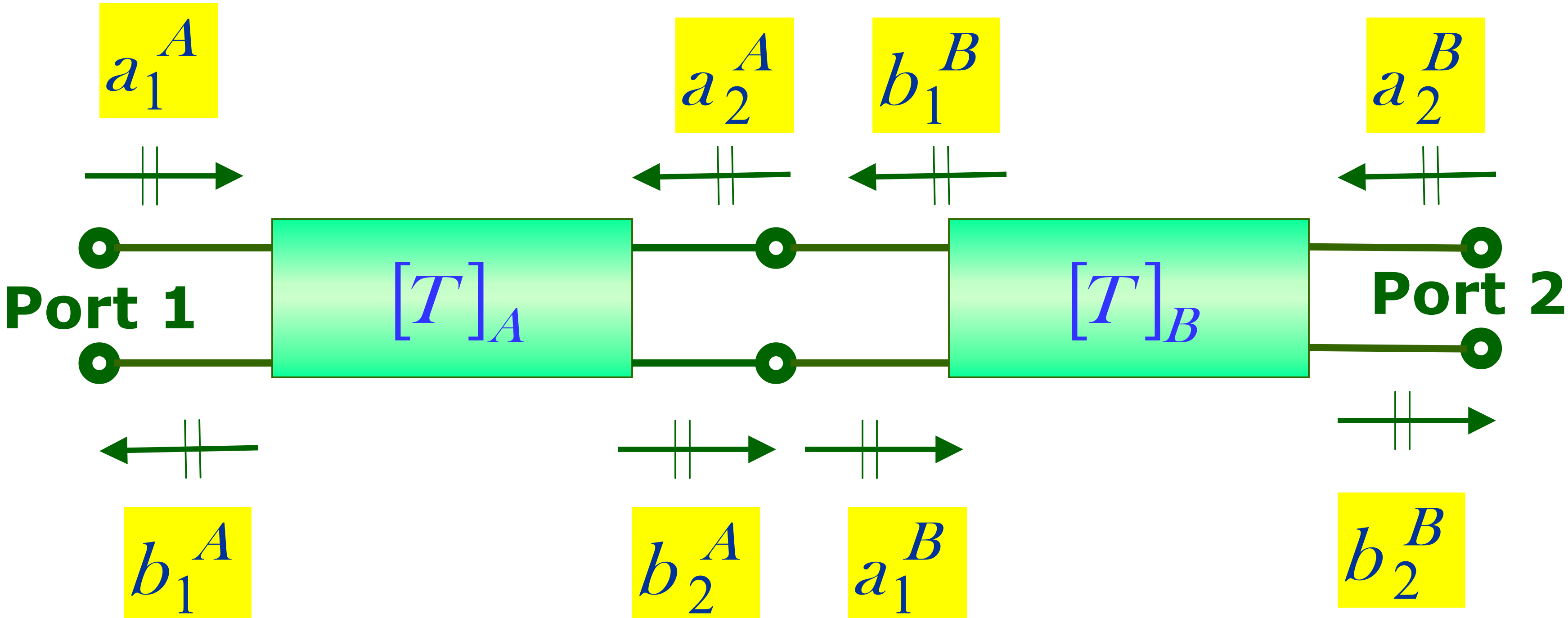
Chain Scattering Matrix

- To extend the concept of the S-parameter presentation to cascaded network, it is more efficient to rewrite the power wave expressions arranged in terms of input and output ports. This results in the chain scattering matrix notation. That is,

$$\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} b_2 \\ a_2 \end{Bmatrix} \qquad (28)$$

- It is immediately seen that cascading of two dual-port networks becomes a simple multiplication.

Chain Scattering Matrix



Cascading of two networks A and B

Chain Scattering Matrix

If network A is described by

$$\begin{Bmatrix} a_1^A \\ b_1^A \end{Bmatrix} = \begin{bmatrix} T_{11}^A & T_{12}^A \\ T_{21}^A & T_{22}^A \end{bmatrix} \begin{Bmatrix} b_2^A \\ a_2^A \end{Bmatrix} \quad (29)$$

And network B by

$$\begin{Bmatrix} a_1^B \\ b_1^B \end{Bmatrix} = \begin{bmatrix} T_{11}^B & T_{12}^B \\ T_{21}^B & T_{22}^B \end{bmatrix} \begin{Bmatrix} b_2^B \\ a_2^B \end{Bmatrix} \quad (30)$$

Chain Scattering Matrix

$$\begin{Bmatrix} b_2^A \\ a_2^A \end{Bmatrix} = \begin{Bmatrix} a_1^B \\ b_1^B \end{Bmatrix} \quad (31)$$

Thus, for the combined system, we conclude

$$\begin{Bmatrix} a_1^A \\ b_1^A \end{Bmatrix} = \begin{bmatrix} T_{11}^A & T_{12}^A \\ T_{21}^A & T_{22}^A \end{bmatrix} \begin{bmatrix} T_{11}^B & T_{12}^B \\ T_{21}^B & T_{22}^B \end{bmatrix} \begin{Bmatrix} b_2^B \\ a_2^B \end{Bmatrix} \quad (31)$$

Chain Scattering Matrix

The conversion from S-matrix to the chain matrix notation is similar as described before.

$$T_{11} = \left.\frac{a_1}{b_2}\right|_{a_2=0} = \frac{a_1}{S_{21}a_1} = \frac{1}{S_{21}} \qquad (32)$$

$$T_{12} = -\frac{S_{22}}{S_{21}} \qquad (33)$$

$$T_{21} = \frac{S_{11}}{S_{21}} \qquad (34)$$

$$T_{22} = \frac{-\left(S_{11}S_{22} - S_{12}S_{21}\right)}{S_{21}} = \frac{-\Delta S}{S_{21}} \qquad (35)$$

Chain Scattering Matrix

Conversely, when the chain scattering parameters are given and we need to convert to S-parameters, we find the following relations:

$$S_{11} = \left.\frac{b_1}{a_2}\right|_{a_2=0} = \frac{T_{21}b_2}{T_{11}b_2} = \frac{T_{21}}{T_{11}} \qquad (36)$$

$$S_{12} = \frac{\left(T_{11}T_{22} - T_{12}T_{21}\right)}{T_{11}} = \frac{\Delta T}{T_{11}} \qquad (37)$$

$$S_{21} = \frac{1}{T_{11}} \qquad (38)$$

$$S_{22} = -\frac{T_{12}}{T_{11}} \qquad (39)$$

Conversion between Z- and S-Parameters

To find the conversion between the S-parameters and the Z-parameters, let us begin with defining S-parameters relation in matrix notation

$$\{b\} = [S]\{a\} \quad (40)$$

Multiplying by $\sqrt{Z_\circ}$ **gives**

$$\sqrt{Z_\circ}\{b\} = \{V^-\} = \sqrt{Z_\circ}[S]\{a\} = [S]\{V^+\} \quad (41)$$

Adding $\{V^+\} = \sqrt{Z_\circ}\{a\}$ **to both sides results in**

$$\{V\} = [S]\{V^+\} + \{V^+\} = ([S] + [E])\{V^+\} \quad (42)$$

Conversion between Z- and S-Parameters

To compare this form with the impedance expression

$$\{V\} = [Z]\{I\}$$

We have to express $\{V^+\}$ in term of $\{I\}$. Subtract $[S]\{V^+\}$ from both sides of

$$\{V^+\} = \sqrt{Z_\circ}\{a\}$$

$$\{V^+\} - [S]\{V^+\} = \sqrt{Z_\circ}(\{a\} - \{b\}) = Z_\circ\{I\} \qquad (43)$$

$$\{V^+\} = Z_\circ([E] - [S])^{-1}\{I\} \qquad (44)$$

Conversion between Z- and S-Parameters

Substituting (44) into (42) yields

$$\{V\} = ([S]+[E])\{V^+\} = Z_\circ([S]+[E])([E]-[S])^{-1}\{I\} \quad (45)$$

or

$$[Z] = Z_\circ([S]+[E])([E]-[S])^{-1} \quad (46)$$

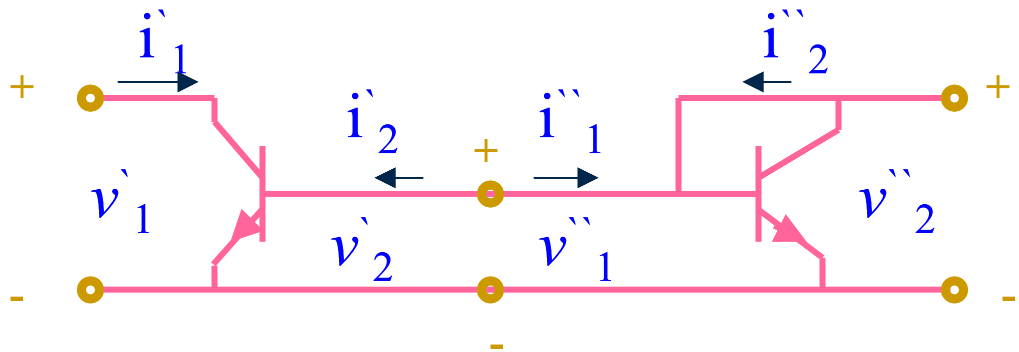
Explicitly

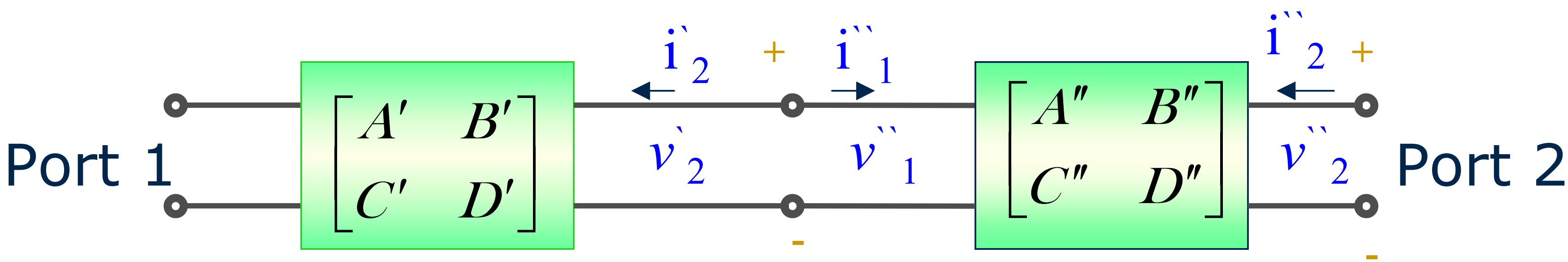
$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = Z_\circ \begin{bmatrix} 1+S_{11} & S_{12} \\ S_{21} & 1+S_{22} \end{bmatrix} \begin{bmatrix} 1-S_{11} & -S_{12} \\ -S_{21} & 1-S_{22} \end{bmatrix}^{-1}$$

$$= \frac{Z_\circ \begin{bmatrix} 1+S_{11} & S_{12} \\ S_{21} & 1+S_{22} \end{bmatrix}}{(1-S_{11})(1-S_{22})-S_{21}S_{12}} \begin{bmatrix} 1-S_{22} & S_{12} \\ S_{21} & 1-S_{11} \end{bmatrix} \quad (47)$$

ABCD parameter representation

- Very useful when cascading networks

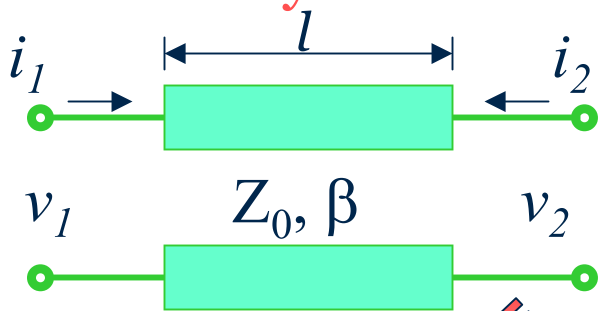




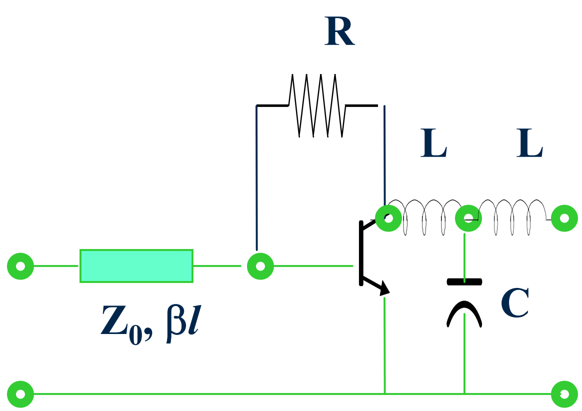
$$\begin{Bmatrix} v_1 \\ i_1 \end{Bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \begin{Bmatrix} v_2'' \\ -i_2'' \end{Bmatrix}$$

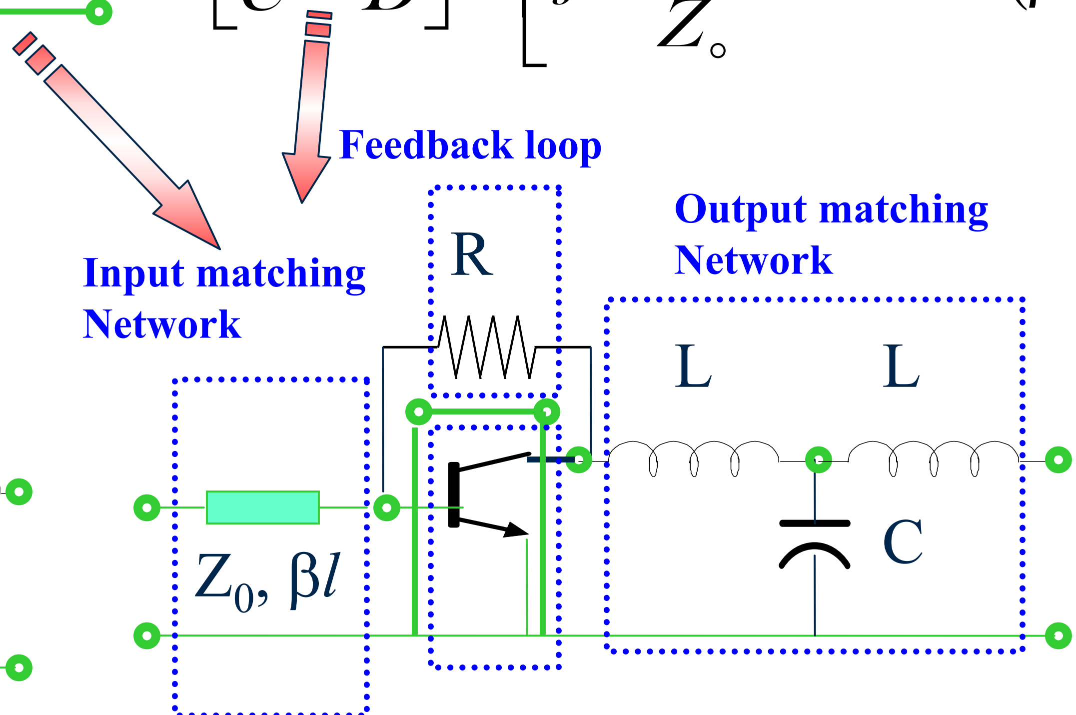
ABCD parameter representation

- ABCD very useful for T.L.



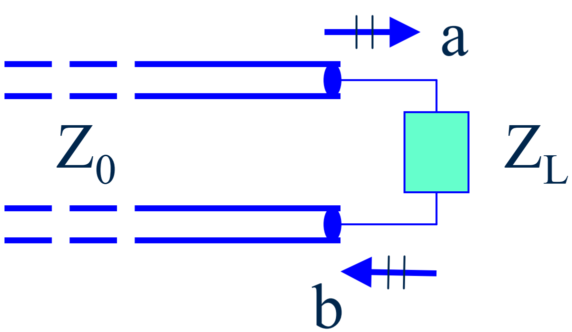
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos(\beta\ell) & jZ_\circ \sin(\beta\ell) \\ j\dfrac{\sin(\beta\ell)}{Z_\circ} & \cos(\beta\ell) \end{bmatrix}$$

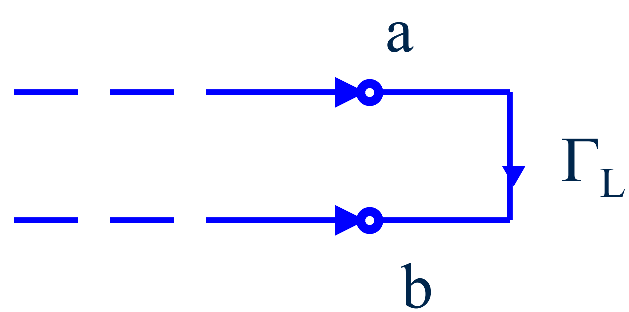




Signal Flow Computations

Complicated networks can be efficiently analyzed in a manner identical to signals and systems and control.





in general

Signal Flow Graphs

Basic Rules:

We'll follow certain rules when we build up a network flow graph.

1. Each variable, a1, a2, b1, and b2 will be designated as a node.

2. Each of the S-parameters will be a branch.

3. Branches enter dependent variable nodes, and emanate from the independent variable nodes.

4. In our S-parameter equations, the reflected waves b1 and b2 are the dependent variables and the incident waves a1 and a2 are the independent variables.

5. Each node is equal to the sum of the branches entering it.

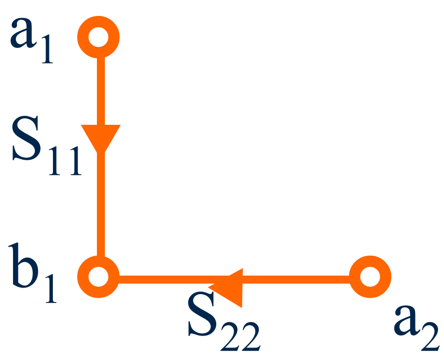
Signal Flow Graphs

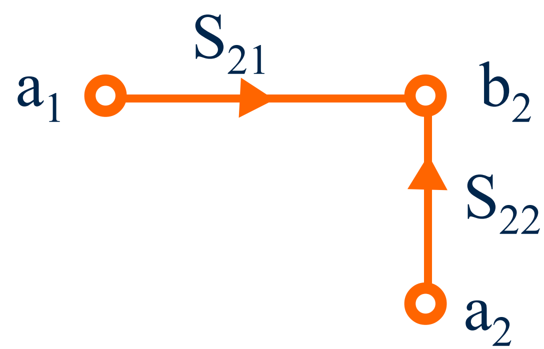
Let’s apply these rules to the two S-parameters equations

$$b_1 = S_{11}a_1 + S_{12}a_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2$$

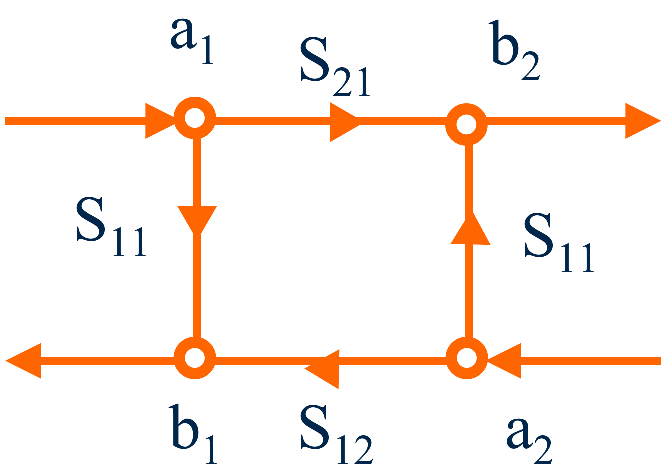
First equation has three nodes: b_1, a_1, and a_2. b_1 is a dependent node and is connected to a_1 through the branch S_{11} and to node a_2 through the branch S_{12}. The second equation is similar.





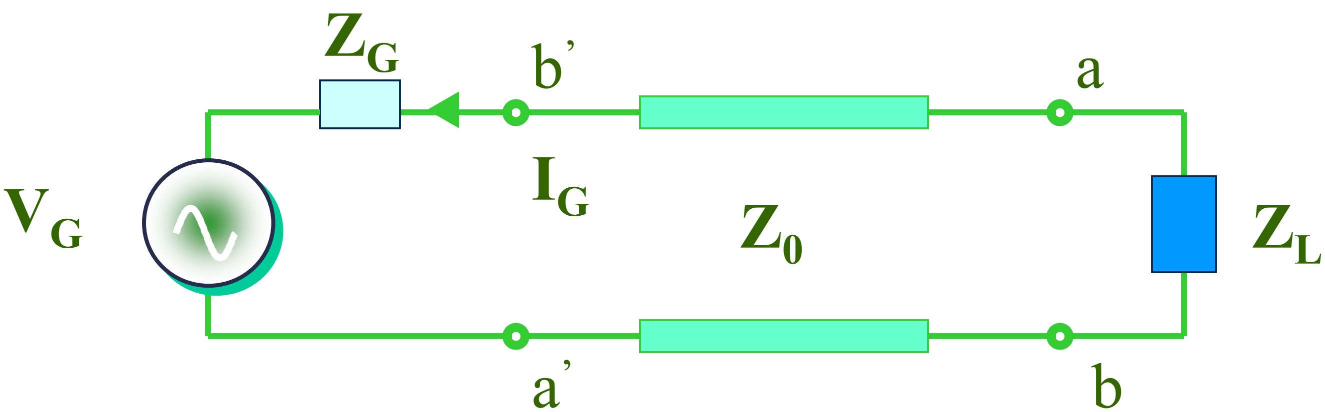
Signal Flow Graphs

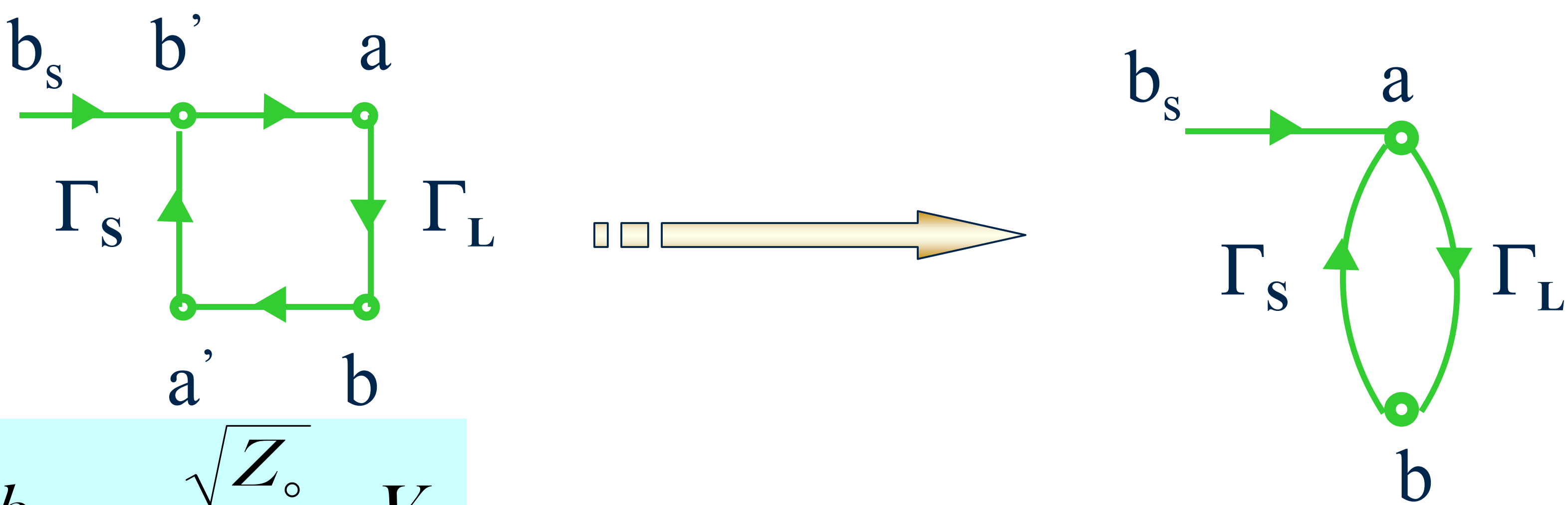
Complete Flow Graph for 2-Port



The relationship between the traveling waves is now easily seen. We have a_1 incident on the network. Part of it transmits through the network to become part of b_2. Part of it is reflected to become part of b_1. Meanwhile, the a_2 wave entering port two is transmitted through the network to become part of b_1 as well as being reflected from port two as part of b_2. By merely following the arrows, we can tell what's going on in the network. This technique will be all the more useful as we cascade networks or add feedback paths.

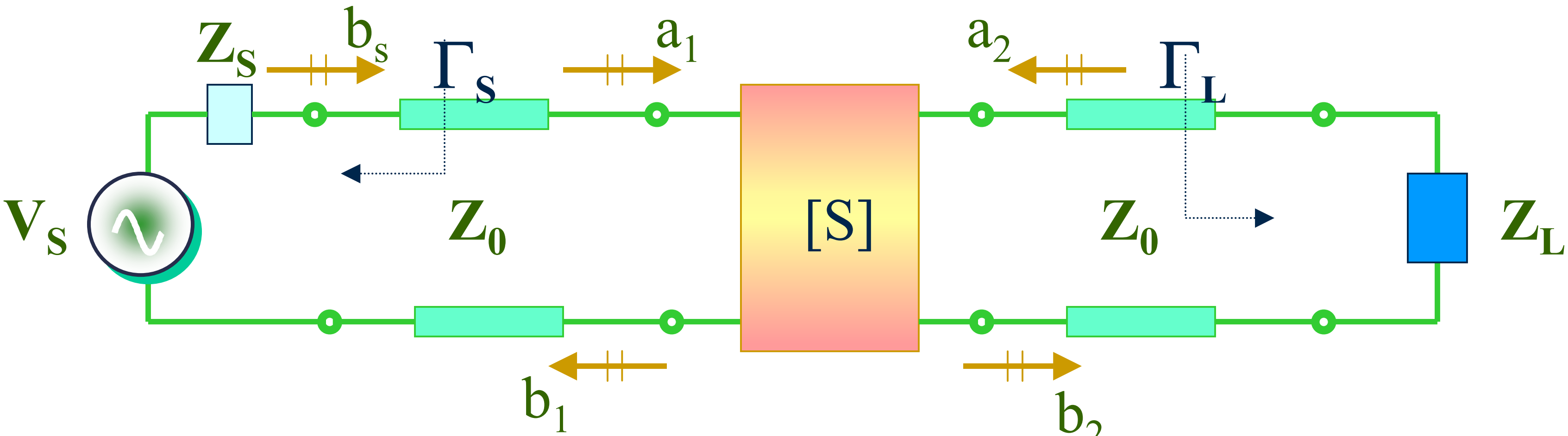
Arrangement for Signal Flow Analysis

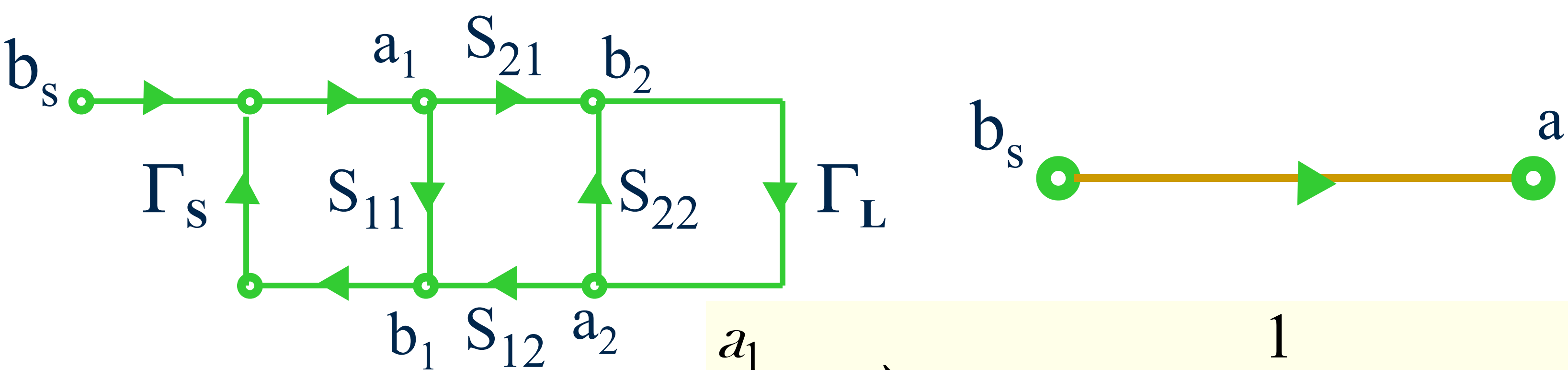




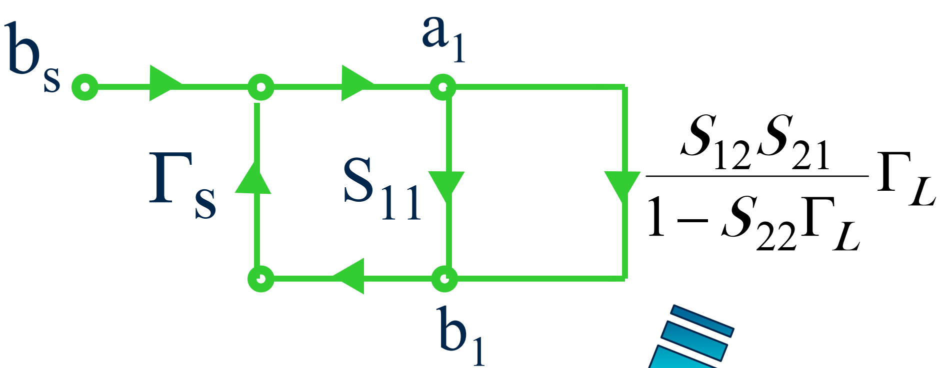
$$b_s = \frac{\sqrt{Z_\circ}}{Z_G + Z_\circ} V_G$$

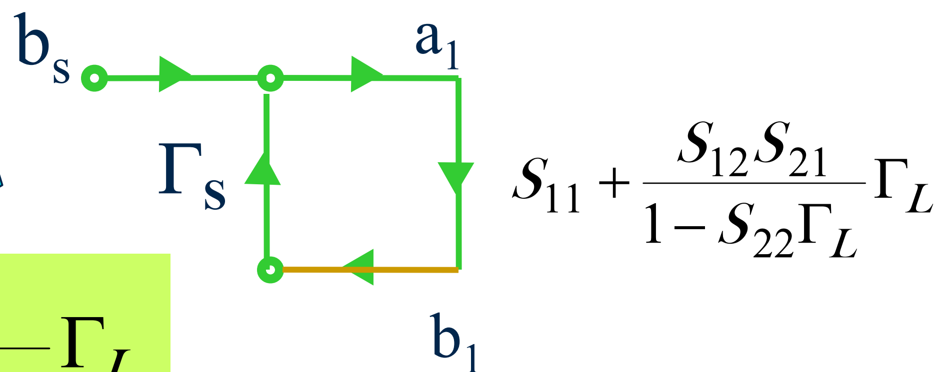
Analysis of Most Common Circuit





$$\frac{a_1}{b_s} \rightarrow \frac{1}{1-\left(S_{11}+\dfrac{S_{12}S_{21}}{1-S_{22}\Gamma_L}\Gamma_L\right)\Gamma_S}$$



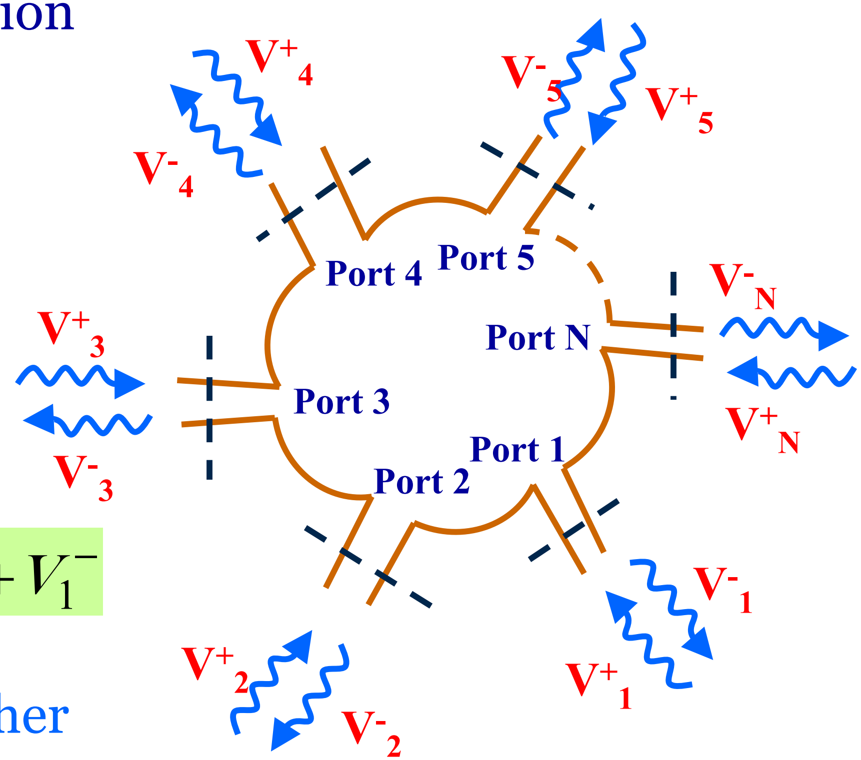


$$\boxed{\Gamma_{in}=\frac{b_1}{a_1}=S_{11}+\frac{S_{12}S_{21}}{1-S_{22}\Gamma_L}\Gamma_L}$$

Note: Only $\Gamma_L = 0$ ensures that S_{11} can be measured.

Scattering Matrix

The scattered-wave amplitudes are linearly related to the incident wave amplitudes. Consider the N port junction



If the only incident wave is V^+_1 then

$$V_1^- = S_{11} V_1^+$$

S_{11} is the reflection coefficient

The total voltage is port 1 is $V_1 = V_1^+ + V_1^-$

Waves will also be scattered out of other ports. We will have

$$V_n^- = S_{n1} V_n^+ \quad n = 2,3,4,\ldots N$$

Scattering Matrix

If all ports have incident wave then

$$\begin{bmatrix} V_1^- \\ V_2^- \\ \cdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1N} \\ S_{21} & S_{22} & S_{23} & \cdots & S_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{N1} & S_{N2} & S_{N3} & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \cdots \\ V_N^+ \end{bmatrix}$$

or

$$\left[V^-\right] = [S]\left[V^+\right]$$

$[S]$ is called the scattering matrix $\quad S_{ij} = \frac{V_i^-}{V_j^+} \;\; for \;\; V_k^+ = 0 \, (k \neq j)$

Scattering Matrix

If we choose the equivalent Z_o equal to 1 then the incident power is given by

$$\frac{1}{2}\left|V_n^+\right|^2$$

and the scattering will be symmetrical. With this choice

$$V = V^+ + V^-, I = I^+ + I^-$$

$$and$$

$$V^+ = \frac{1}{2}(V + I)$$

$$V^- = \frac{1}{2}(V - I)$$

Scattering Matrix

V^+ and V^- are the variables in the scattering matrix formulation; but they are linear combination of V and I.

Other normalization are

$$\nu = \frac{V}{\sqrt{Z_\circ}} \qquad i = \frac{I}{\sqrt{Z_\circ}}$$

Just as in the impedance matrix there are several properties of the scattering matrix we want to consider.

1. A shift of the reference planes
2. S matrix for reciprocal devices
3. S matrix for the lossless devices

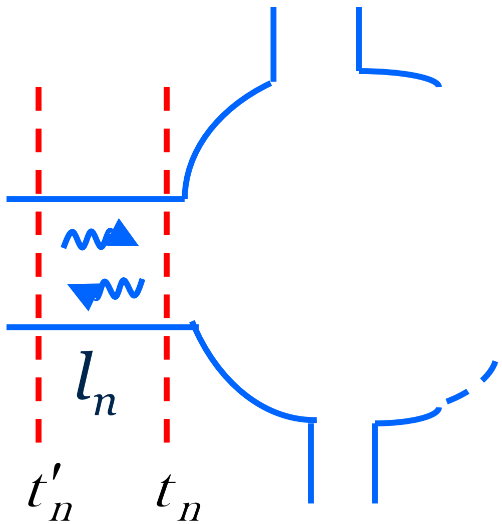
Scattering Matrix

Shift in the reference planes

Consider the following network, where t_n is the original location of the reference plane, and t'_n in the new location of the reference plane.

The electrical length between t_n and t'_n is

$\theta = \beta_n \ell_n$.



S_{mn} $m \neq n$ *must be multiplied by* $e^{-j\theta_n}$

S_{nn} *must be multiplied by* $e^{-j2\theta_n}$

Why is this a factor of 2?

Scattering Matrix

$$V_n'^{+} = V_n^{+} e^{j\theta_n}$$

$$V_n'^{-} = V_n^{-} e^{-j\theta_n}$$

$$[S] = \begin{bmatrix} e^{-j\theta_1} & & & & \\ & e^{-j\theta_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{-j\theta_N} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ S_{N1} & S_{N2} & \cdots & S_{nn} \end{bmatrix} \begin{bmatrix} e^{-j\theta_1} & & & & \\ & e^{-j\theta_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & e^{-j\theta_N} \end{bmatrix}$$

Scattering Matrix

Consider a 2×2 scattering matrix where two reference planes are shifted.

$$[S'] = \begin{bmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{bmatrix}$$

$$[S'] = \begin{bmatrix} S_{11}e^{-j\theta_1} & S_{12}e^{-j\theta_1}e^{-j\theta_2} \\ S_{21}e^{-j\theta_1}e^{-j\theta_2} & S_{22}e^{-j\theta_2} \end{bmatrix}$$

$$[S'] = \begin{bmatrix} S_{11}e^{-j2\theta_1} & S_{21}e^{-j\theta_1}e^{-j\theta_2} \\ S_{12}e^{-j\theta_1}e^{-j\theta_2} & S_{22}e^{-j2\theta_2} \end{bmatrix}$$

Scattering Matrix

Proof of Symmetry of Scattering Matrix

For a reciprocal junction $S_{mn} = S_{nm}$ $m \neq n$ provided that $Z_0=1$

$$P = \frac{1}{2}\left|V_n^+\right|^2$$ **for all modes.**

$$V_n = V_n^+ + V_n^- \quad and \quad I_n = I_n^+ - I_n^-$$

$$\therefore I_n = V_n^+ - V_n^- \quad (Z_\circ = 1)$$

$$[V] = [V^+] + [V^-]$$

$$= [Z][I]$$

$$= [Z][V^+] - [Z][V^-]$$

Scattering Matrix

Defining the unit matrix

$$[u]=\begin{bmatrix}1 & & 0\\ & 1 & \\ & & . & \\ 0 & & 1\end{bmatrix}$$

$$[V^+]+[V^-]=[Z][V^+]-[Z][V^-]$$

Rearrange and factor

$$([Z]+[U])[V^-]=([Z]-[U])[V^+]$$

or

$$[V^-]=([Z]+[U])^{-1}([Z]-[U])[V^+]$$

but $[V^-]=[S][V^+]$ $\quad\therefore [S]=([Z]+[U])^{-1}([Z]-[U])$

Scattering Matrix

Loss-less Junction

The total power leaving a junction must be equal to the total power entering the junction power.

$$\sum_{n=1}^{N} \left| V_n^- \right|^2 = \sum_{n=1}^{N} \left| V_n^+ \right|^2$$

but

$$V_n^- = \sum_{i=1}^{N} S_{ni} V_i^+$$

so that

$$\sum_{n=1}^{N} \left| \sum_{i=1}^{N} S_{ni} V_i^+ \right|^2 = \sum_{n=1}^{N} \left| V_n^+ \right|^2$$

Scattering Matrix

Loss-less Junction

V^+_n are all independent incident voltages, so we choose V^+_n =0 except for n=i

$$\sum_{n=1}^{N} \left| \sum_{i=1}^{N} S_{ni} V_i^+ \right|^2 = \sum_{n=1}^{N} \left| V_n^+ \right|^2$$

$$\sum_{n=1}^{N} \left| S_{ni} V_i^+ \right|^2 = \left| V_n^+ \right|^2$$

$$\sum_{n=1}^{N} \left| S_{ni} \right|^2 = \sum_{n=1}^{N} S_{ni} S_{ni}^* = 1 \quad \forall i.$$

$$\begin{bmatrix} S_{1i} & S_{2i} & \dots & S_{Ni} \end{bmatrix} \begin{bmatrix} S_{1i}^* \\ S_{2i}^* \\ \dots \\ S_{Ni}^* \end{bmatrix} = 1$$

Scattering Matrix

Loss-less 2 Port Junction

For this case [S] is unitary

$$\therefore \sum_{n=1}^{N} S_{nm} S_{np}^{*} = \delta_{mp}$$

or

$$\left.\begin{aligned} S_{11}S_{11}^{*} + S_{12}S_{12}^{*} &= 1 \\ S_{22}S_{22}^{*} + S_{12}S_{12}^{*} &= 1 \end{aligned}\right\} \Rightarrow |S_{11}| = |S_{22}|$$

$$S_{11}S_{12}^{*} + S_{12}S_{22}^{*} = 0$$

The magnitude of the input and output ports are equal in magnitude. Also

$$|S_{12}| = \sqrt{1 - |S_{11}|^{2}}$$

Scattering Matrix

If we know $\boxed{|S_{11}|}$ then we can obtain $\boxed{|S_{12}|}$ and $\boxed{|S_{22}|}$.

Note: The fraction of power reflected at terminal *t*, is

$$\frac{P_{refl}}{P_{inc}} = |S_{11}|^2$$

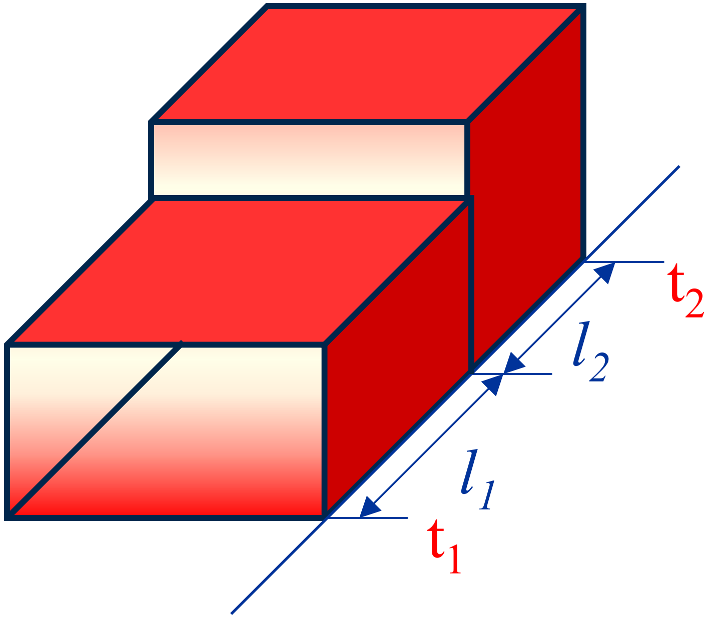
So that the insertion loss due to reflection is

$$IL = 10\,log\left(1 - |S_{11}|^2\right)$$

$$IL = 20\,log|S_{12}|$$

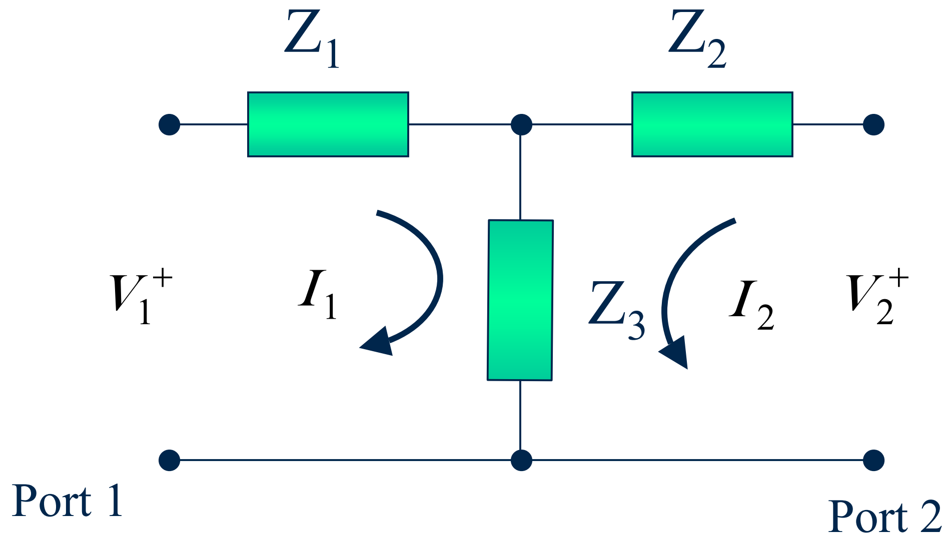
Scattering Matrix

Example: two-port network



Assume TE_{10} modes at t_1 and t_2

Equivalent Circuit



Apply KVL:

$$V_1 = Z_1 I_1 + Z_3 I_1 + Z_3 I_2$$

$$V_2 = Z_2 I_2 + Z_3 I_2 + Z_3 I_1$$

Scattering Matrix

If

$$Z_3 = Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

$$Z_1 = Z_{11} - Z_{12}$$

$$Z_2 = Z_{22} - Z_{12}$$

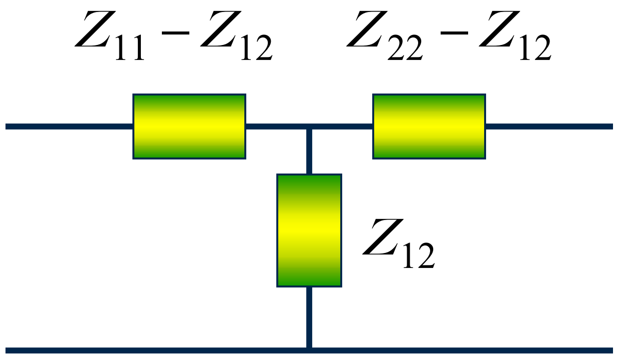
Then we have

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{22} I_2 + Z_{12} I_2$$

and

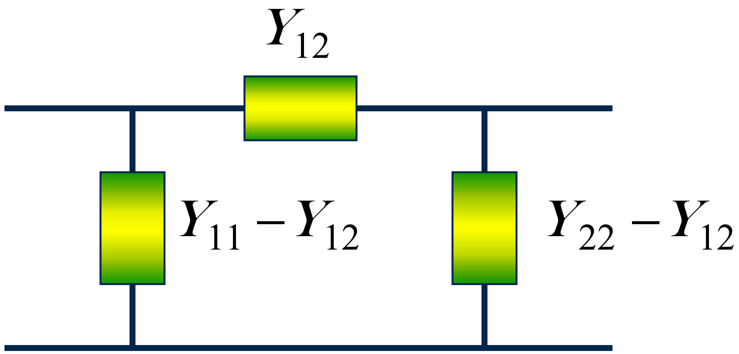
$$[V] = [Z][I]$$



Scattering Matrix

This can be transformed into an admittance matrix

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$



Scattering Matrix

Traveling Wave:

$$V^{+}=Ae^{-\delta x},V^{-}=Ae^{\delta x}$$

$$V(x)=V^{+}(x)+V^{-}(x)$$

Similarly for current:

$$I(x)=I^{+}(x)-I^{-}(x)=\frac{V^{+}(x)}{Z_{\circ}}-\frac{V^{-}(x)}{Z_{\circ}}$$

Reflection Coefficient:

$$\Gamma(x)=\frac{V^{-}(x)}{V^{+}(x)}$$

Scattering Matrix

Introduce “normalized” variables:

$$v(x) = V(x)/\sqrt{Z_\circ}\,,\ i(x) = \sqrt{Z_\circ}\,I(x)$$

So that

$$v(x) = a(x) + b(x) \quad i(x) = a(x) - b(x) \quad \text{and} \quad b(x) = \Gamma(x)a(x)$$

This defines a single port network. What about 2-port?

2-port

$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

Scattering Matrix

Each reflected wave (b_1,b_2) has two contributions: one from the incident wave at the same port and another from the incident wave at the other port.

How to calculate S-parameters?

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0}$$ Input reflected coefficient with output matched.

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a1=0}$$ Reverse transmission coefficient with input matched.

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$ Transmission coefficient with output matched.

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0}$$ Output reflected coefficient with input matched.

Scattering Matrix

Generalized Scattering Matrix:

We define it via [b]=[S][a]. S-matrix depends on the choice of normalized impedance. Usually 50 Ω, but can be anything and can even be complex!

Calculating S_{ij}:

$$S_{ij} = \frac{b_i}{a_i}\bigg|_{a_k=0, k\neq i, k=1,\ldots n} = \frac{V_i - Z^*_{\circ,i} I_i}{V_i + Z^*_{\circ,i} I_i} = \frac{Z_i - Z^*_{\circ,i}}{Z_i + Z_{\circ,i}}$$

Which is input reflected coefficient with all other ports matched.

$$S_{ki} = \frac{bk}{a_i}\bigg|_{a_k=0, k\neq i, k=1,\ldots n}$$

is equal to transducer power gain from i to k with ports other than i matched.