

RF and Microwave Physics

Lecture 2

Maxwell's equations

Grad, div and curl

Wave equations

Plane Waves

Boundary conditions

Maxwell's Equations

The general form of the time-varying Maxwell's equations can be written in differential form as:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

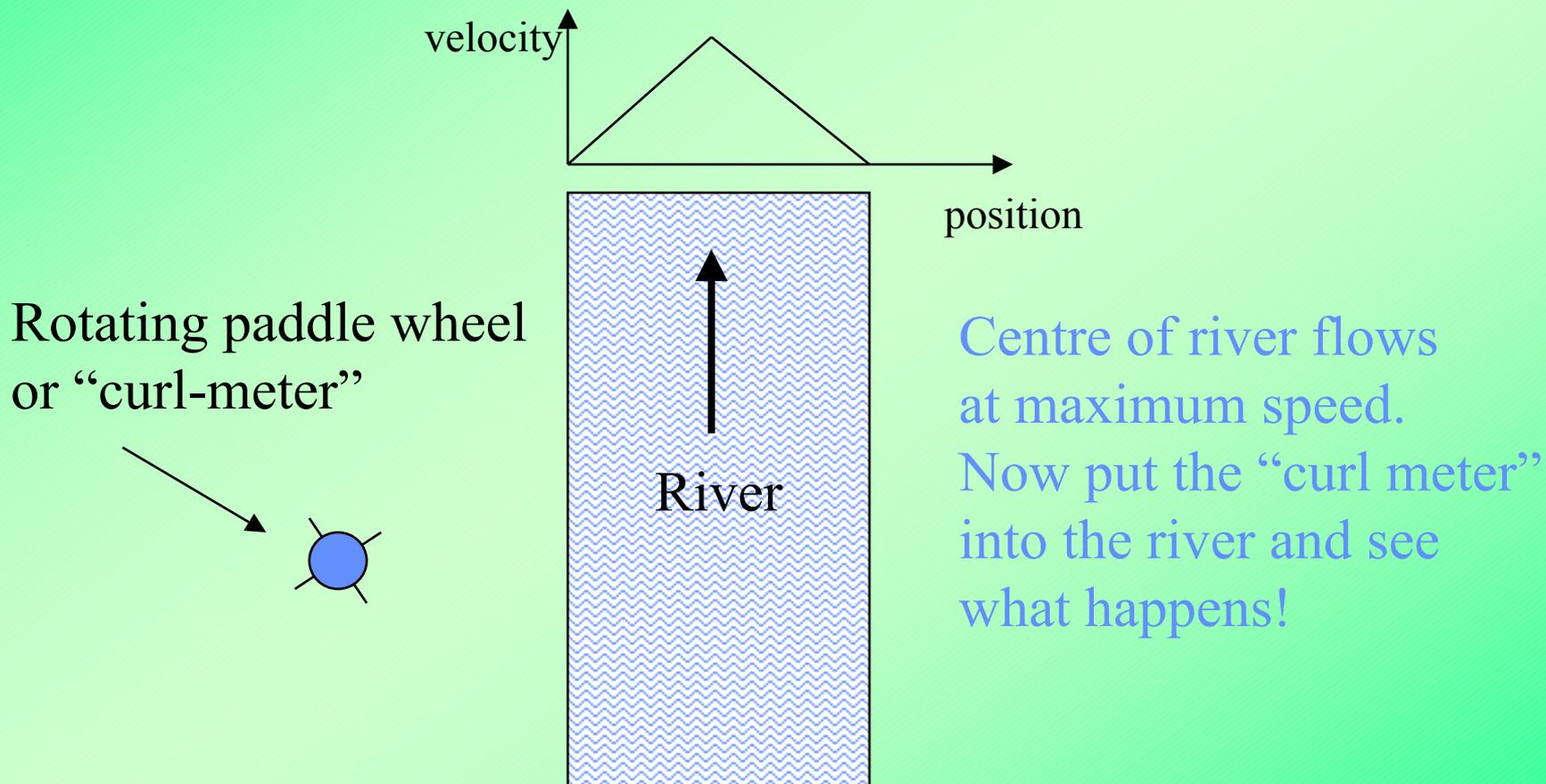
$$\nabla \cdot \mathbf{B} = 0$$

What is Maxwell's theory?

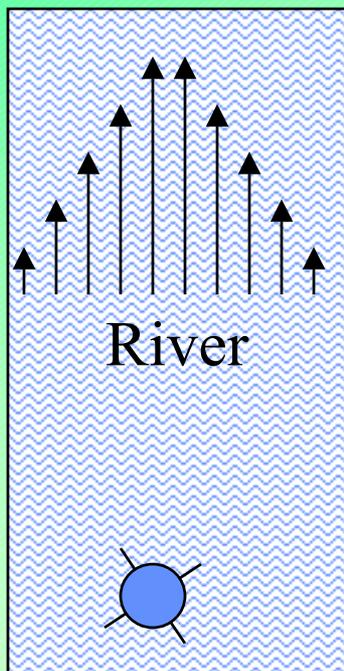
**“I cannot give any clearer or briefer answer than the following:
Maxwell's theory is the system of Maxwell's equations”**

**-Heinrich Rudolph Hertz
(1857-94)**

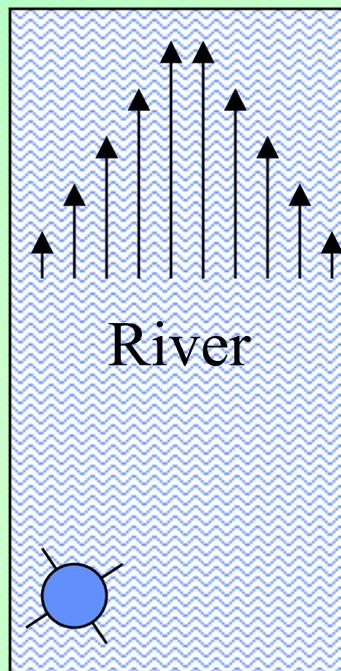
Physical interpretation of curl



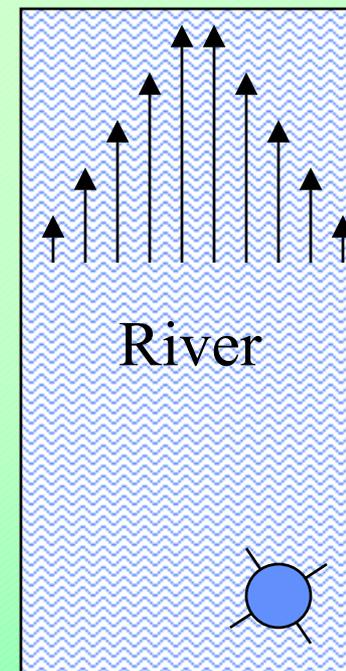
Three different positions (centre left and right)



No rotation!



Anti-clockwise
rotation.



Clockwise
rotation.

Conclusions

- Curl-meter only indicates rotation if there is non-uniformity in the vector field.
- The amount of rotation is proportional to the degree of non-uniformity.
- The rotation cannot be described just with a scalar. Direction should also be given. Rotation is therefore a vector quantity.

Curl therefore describes the variation across the field. The above physical interpretation is concerned with only one dimension or one component. For electromagnetic fields we must imagine the concept applying to three dimensions.

A few other fundamental relationships

$$\mathbf{J} = \sigma \mathbf{E}$$

"Ohm's law"

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

"continuity equation"

$$\left. \begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E} \\ \mathbf{B} &= \mu \mathbf{H} \end{aligned} \right\}$$

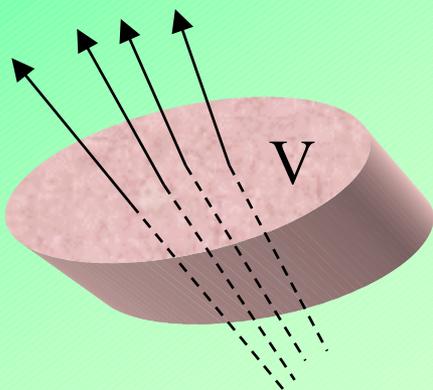
"constitutive relationships"

Note this well!!!

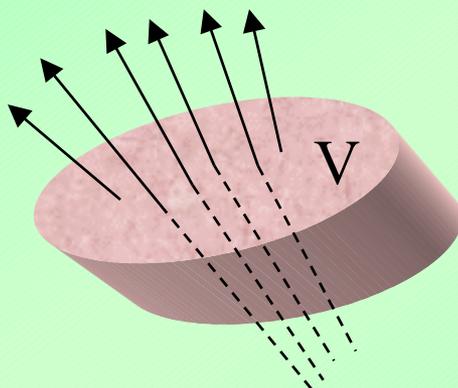
here $\varepsilon = \varepsilon_0 \varepsilon_r$ (permittivity) and $\mu = \mu_0 \mu_r$ (permeability)

with $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, $\mu_0 = 4\pi \times 10^{-7}$ H/m

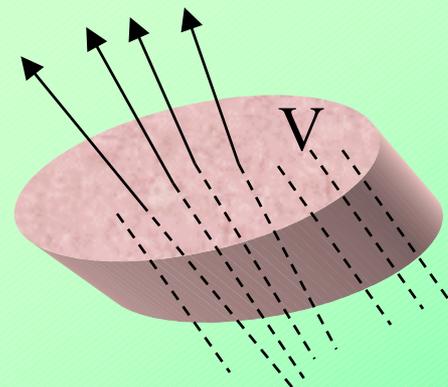
Physical interpretation of divergence



Flux in = flux out
so **no sources or sinks** inside V .



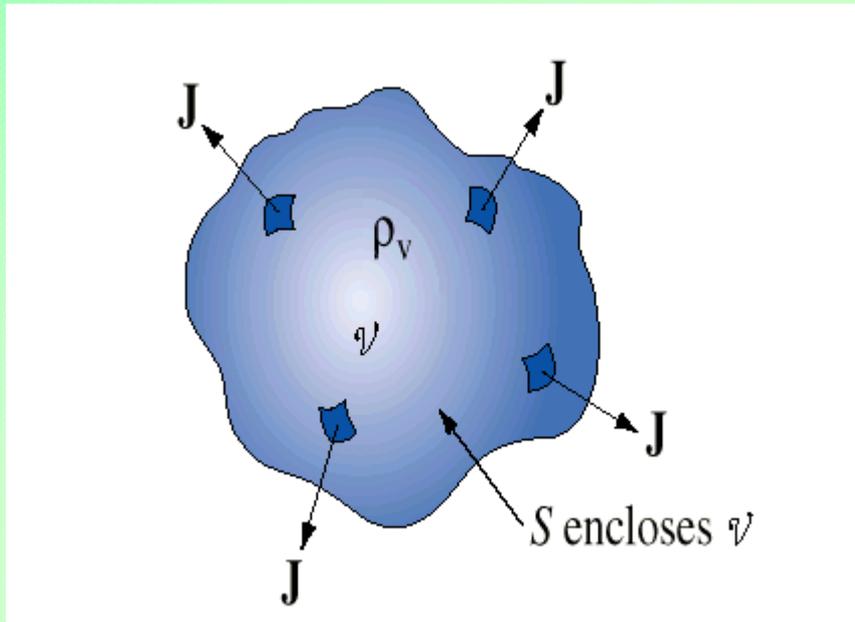
Flux out $>$ flux in
Positive
divergence.
Must be a **source**
inside V .



Flux out $<$ flux in
Negative
divergence.
Must be a **sink** or
drain inside V .

Example

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_V}{\partial t}$$



The current density flux flowing out of the closed surface equals the rate of decrease of the positive charge density.

Maxwell's Equations

- All (macroscopic) electromagnetic phenomena can be explained in terms of Maxwell's equations, the continuity equation and the Lorentz force equation.
- We have only listed the differential form of Maxwell's equations. There are corresponding integral forms of these equations as well.
- We also note that Maxwell's equations are not independent. In fact, the two Divergence equations can be derived from the two Curl equations.

Maxwell's Equations

- In vacuum,

$$\nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{B} = 0$$

SI

We use SI

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

CGS

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$$

- Try eliminating \mathbf{B}

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \quad \frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Do you know the "BAC-CAB" rule?

Wave Equations

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Very useful rule
Prove it!

- Using this rule,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - (\nabla \cdot \nabla)\mathbf{E} = -\nabla^2 \mathbf{E}$$

= 0

- We get $\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}$

Wave equations
for the EM waves
in free space

- Similarly, we can derive

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B}$$

- They look just like the 3-D wave equation from 2 weeks ago
 - We know the solutions

Plane Waves

- Solutions must be plane waves

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \quad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \quad \omega = ck$$

- \mathbf{E}_0 and \mathbf{B}_0 are not completely free
- Must satisfy all of Maxwell's equations

- $\nabla \cdot \mathbf{E} = 0 \Rightarrow \mathbf{k} \cdot \mathbf{E} = 0$

- $\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{k} \cdot \mathbf{B} = 0$

E and B are perpendicular to k

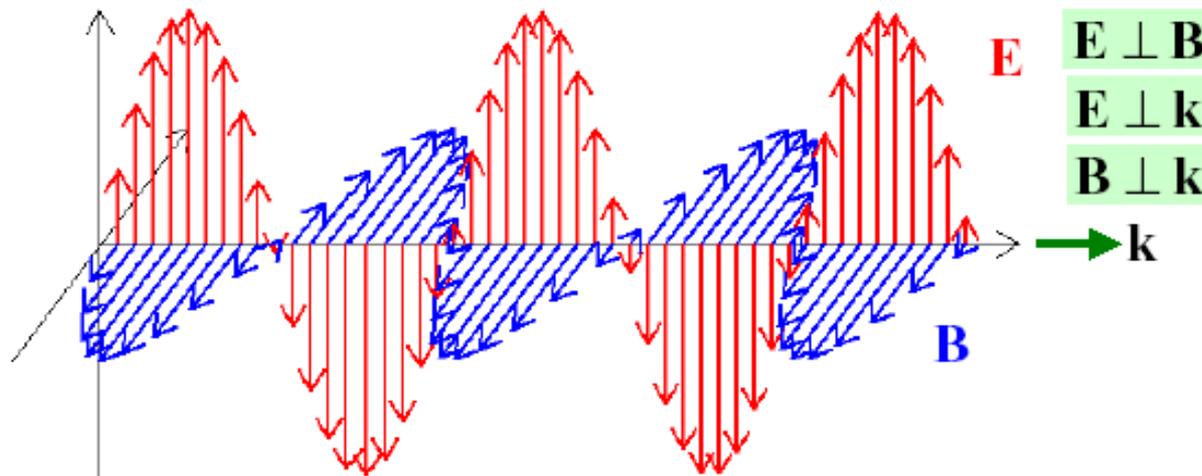
- $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$

- $\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \mathbf{k} \times \mathbf{B} = -\frac{\omega}{c^2} \mathbf{E}$

E and B are perpendicular to each other

Transverse Waves

- EM waves in free space is **transverse**



- From $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$ and $\omega = ck \Rightarrow |\mathbf{E}| = c |\mathbf{B}|$
- If you want \mathbf{H} ,

$$|\mathbf{H}| = \frac{|\mathbf{B}|}{\mu_0} = \frac{|\mathbf{E}|}{c\mu_0} = \frac{|\mathbf{E}|}{Z_0}$$

Vacuum impedance (377Ω)

$|\mathbf{E}| = |\mathbf{B}|$ if we were using CGS

Maxwell's Equations

- Now we go back to Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B} &= \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \end{aligned}$$

- Movement of the charges in matter \rightarrow Current $\mathbf{J} = qn_0\mathbf{v}$

- We assumed $\mathbf{x} = \frac{q\mathbf{E}}{k_s} \rightarrow \mathbf{v} = \frac{q}{k_s} \frac{\partial \mathbf{E}}{\partial t}$

- Usual trick with BAC-CAB rule gives us

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\mu_0 q^2 n_0}{k_s} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Simple wave equation

Plane Wave Solution

$$\nabla^2 \mathbf{E} = \left(\frac{1}{c^2} + \frac{\mu_0 q^2 n_0}{k_s} \right) \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \leftarrow \quad \mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$$

- Wave equation reduces to
- Dispersion relation is

$$-k^2 \mathbf{E} = -\omega^2 \left(\frac{1}{c^2} + \frac{\mu_0 q^2 n_0}{k_s} \right) \mathbf{E}$$

$$k^2 = \frac{\omega^2}{c^2} \left(1 + \frac{q^2 n_0}{\epsilon_0 k_s} \right)$$

$$\rightarrow c_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 + \frac{q^2 n_0}{\epsilon_0 k_s}}}$$

$$n = \sqrt{1 + \frac{q^2 n_0}{\epsilon_0 k_s}}$$

Same n_{true}

- We found the same solution
 - We used the short-cut by trusting Maxwell's \mathbf{J} term
 - It can be made even simpler...

Maxwell's Equation

- Take the equation $\nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$

- We are assuming

$$\mathbf{J} = qn_0 \mathbf{v} \quad \mathbf{v} = \frac{q}{k_s} \frac{\partial \mathbf{E}}{\partial t}$$



$$\begin{aligned} \nabla \times \mathbf{B} &= \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\mu_0 q^2 n_0}{k_s} \frac{\partial \mathbf{E}}{\partial t} \\ &= \left(\varepsilon_0 + \frac{q^2 n_0}{k_s} \right) \mu_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

- We could define

$$\varepsilon = \varepsilon_0 + \frac{q^2 n_0}{k_s}$$

$$\nabla \times \mathbf{B} = \varepsilon \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

- We absorbed the \mathbf{J} term into the matter's permittivity ε

- Now it's easy to get $n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{1 + \frac{q^2 n_0}{\varepsilon_0 k_s}}$

Integral form of the equations

$$\nabla \cdot \vec{D} = \rho \quad \Rightarrow \quad \oint_S \vec{D} \cdot d\vec{s} = Q$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \oint_S \vec{B} \cdot d\vec{s} = 0$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \Rightarrow \quad \oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s}$$

Maxwell's equations not independent!

E.g.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \nabla \times \mathbf{E} = 0 = -\frac{\partial}{\partial t} [\nabla \cdot \mathbf{B}]$$

This implies
 $\nabla \cdot \mathbf{B} = 0$

Use the identity

$$\nabla \cdot \nabla \times \vec{A} \equiv 0$$

$$\nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} [\nabla \cdot \mathbf{D}]$$

$$0 = -\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial t}$$

This implies
 $\nabla \cdot \mathbf{D} = \rho$

Wave Equations

In any problem with unknown \mathbf{E} , \mathbf{D} , \mathbf{B} , \mathbf{H} we have 12 unknowns. To solve for these we need 12 scalar equations. Maxwell's equations provide 3 each for the two curl equations. and 3 each for both constitutive relations (difficult task).

Instead we anticipate that electromagnetic fields propagate as waves. Thus if we can find a wave equation, we could solve it to find out the fields directly.

Wave equations

Take the curl of the first Maxwell:

$$\begin{aligned}
 \nabla \times \nabla \times \mathbf{H} &= \nabla \times \mathbf{J} + \nabla \times \frac{\partial}{\partial t} (\epsilon \mathbf{E}) \\
 &= \nabla \times \mathbf{J} + \epsilon \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) \\
 &= \nabla \times \mathbf{J} + \epsilon \frac{\partial}{\partial t} \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) \\
 &= \nabla \times \mathbf{J} - \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}
 \end{aligned}$$

Now use $\nabla \times \nabla \times \mathbf{H} \equiv \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H}$ on the LHS

0

Wave Equations

The result is:

$$\nabla^2 \vec{\mathbf{H}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{H}}}{\partial t^2} = -\nabla \times \vec{\mathbf{J}}$$

Similarly, the same process for the second Maxwell produces

$$\nabla^2 \vec{\mathbf{E}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = \mu \frac{\partial \vec{\mathbf{J}}}{\partial t} + \nabla \frac{\rho}{\epsilon}$$

Note how in both case we have a wave equation (2nd order PDE) for both \mathbf{E} and \mathbf{H} with fields to the **left** of the = sign and sources to the **right**. These two wave equations are completely equivalent to the Maxwell equations.

Solutions to the wave equations

Consider a region of free space ($\sigma = 0$) where there are no sources ($\mathbf{J} = 0$). The wave equations become homogeneous:

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{H} - \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

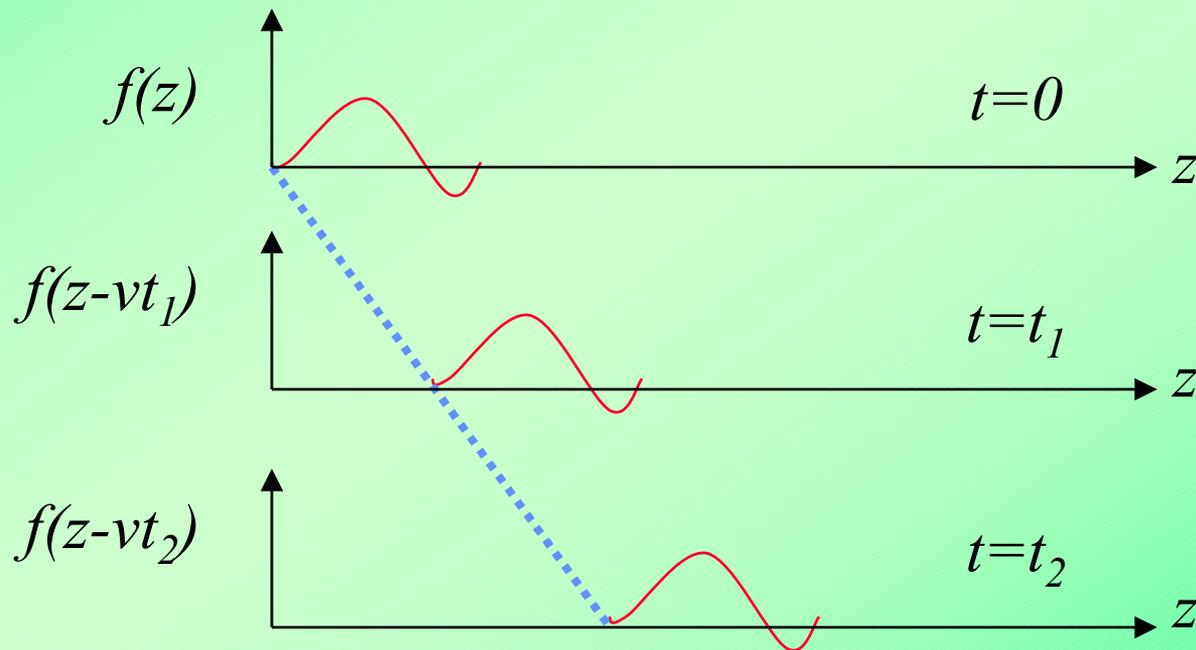
Actually there are 6 equations; we will only consider one component:

e.g. $E_x(z, t)$

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 E_x}{\partial t^2} = 0 \quad \text{where} \quad v^2 = \frac{1}{\mu_0 \epsilon_0} = c^2$$

Solutions to the wave equation

Try a solution of the form $f(z-vt)$ e.g. $\sin[\beta(z-vt)]$. By differentiating twice and substituting back into the scalar wave equation, we find that it satisfies!



Plane Waves

- First treat plane waves in free space.
- Then interaction of plane waves with media.
- We assume time harmonic case, and source free situation.

We require solutions for E and H (which are solutions to the following PDE) in free space

$$\nabla^2 \vec{E} + k_0^2 \vec{E} = 0$$

No potentials here!
(no sources)

Note that this is actually three equations:

$$\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} + k_0^2 E_i = 0 \quad i = x, y, z$$

How do we find a solution?

Usual procedure is to use Separation of Variables (SOV).
Take one component for example E_x .

$$E_x = f(x)g(y)h(z)$$

$$ghf'' + fhg'' + fgh'' + k_0^2 fgh = 0$$

$$\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} + k_0^2 = 0 \quad \longrightarrow \quad \text{Functions of a single variable} \Rightarrow \text{sum} = \text{constant} = -k_0^2$$

$$\frac{f''}{f} = -k_x^2; \quad \frac{g''}{g} = -k_y^2; \quad \frac{h''}{h} = -k_z^2$$

and so

$$k_x^2 + k_y^2 + k_z^2 = k_0^2 \quad \text{with} \quad k_0 = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

Mathematical Solution

We note we have 3 ODEs now.

$$\frac{d^2 f}{dx^2} + k_x^2 f = 0 \quad \text{solution is} \quad f = e^{\pm jk_x x}$$

$$\frac{d^2 g}{dy^2} + k_y^2 g = 0 \quad \text{solution is} \quad g = e^{\pm jk_y y}$$

$$\frac{d^2 h}{dz^2} + k_z^2 h = 0 \quad \text{solution is} \quad h = e^{\pm jk_z z}$$

$$E_x = A e^{\pm j(k_x x + k_y y + k_z z)}$$

But, what does it mean physically?

$$E_x = A e^{\pm j(k_x x + k_y y + k_z z)}$$

This represents the x-component of the travelling wave E-field (like on a transmission line) which is travelling in the direction of the propagation vector, with Amplitude A. The direction of propagation is given by

$$\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$$

Physical interpretation

The solution represents a wave travelling in the $+z$ direction with velocity c . Similarly, $f(z+vt)$ is a solution as well. This latter solution represents a wave travelling in the $-z$ direction.

So generally,

$$E_x(z, t) = f[(x \pm vt)(y \pm vt)(z \pm vt)]$$

In practice, we solve for either E or H and then obtain the other field using the appropriate curl equation.

What about when sources are present? Looks difficult!

Generalize for all components

If we define the normal 3D position vector as:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

then $\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z$

so $E_x = Ae^{-j\vec{k} \cdot \vec{r}}$ ← +sign dropped here

similarly $E_y = Be^{-j\vec{k} \cdot \vec{r}}$

$E_z = Ce^{-j\vec{k} \cdot \vec{r}}$

General expression
for a plane wave

→ $\vec{E} = \vec{E}_0 e^{-j\vec{k} \cdot \vec{r}}$ where $\vec{E}_0 = A\hat{x} + B\hat{y} + C\hat{z}$

Properties of plane waves

For source free propagation we must have $\nabla \cdot \mathbf{E} = 0$. If we satisfy this requirement we must have $\mathbf{k} \cdot \mathbf{E}_0 = 0$. This means that \mathbf{E}_0 is perpendicular to \mathbf{k} .

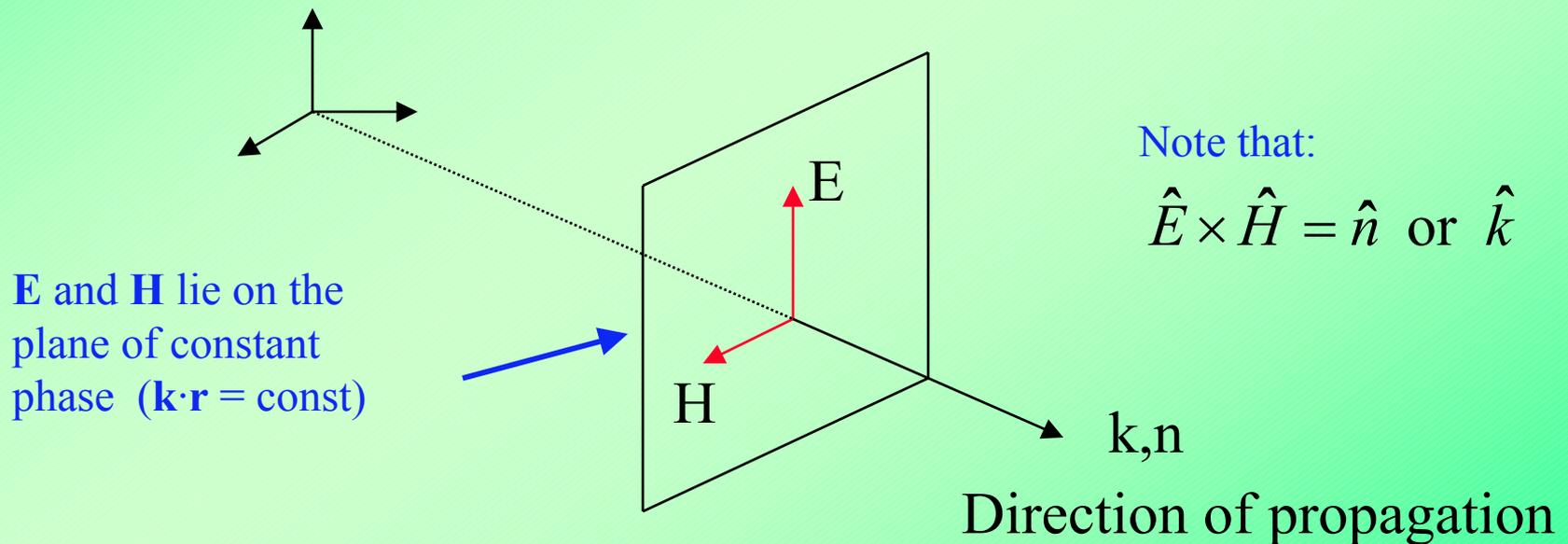
The corresponding expression for \mathbf{H} can be found by substitution of the solution for \mathbf{E} into the $\nabla \times \mathbf{E}$ equation. The result is:

$$\vec{H} = \frac{k_0}{\omega \mu_0} \hat{n} \times \vec{E}$$

Where \mathbf{n} is a unit vector in the \mathbf{k} direction.

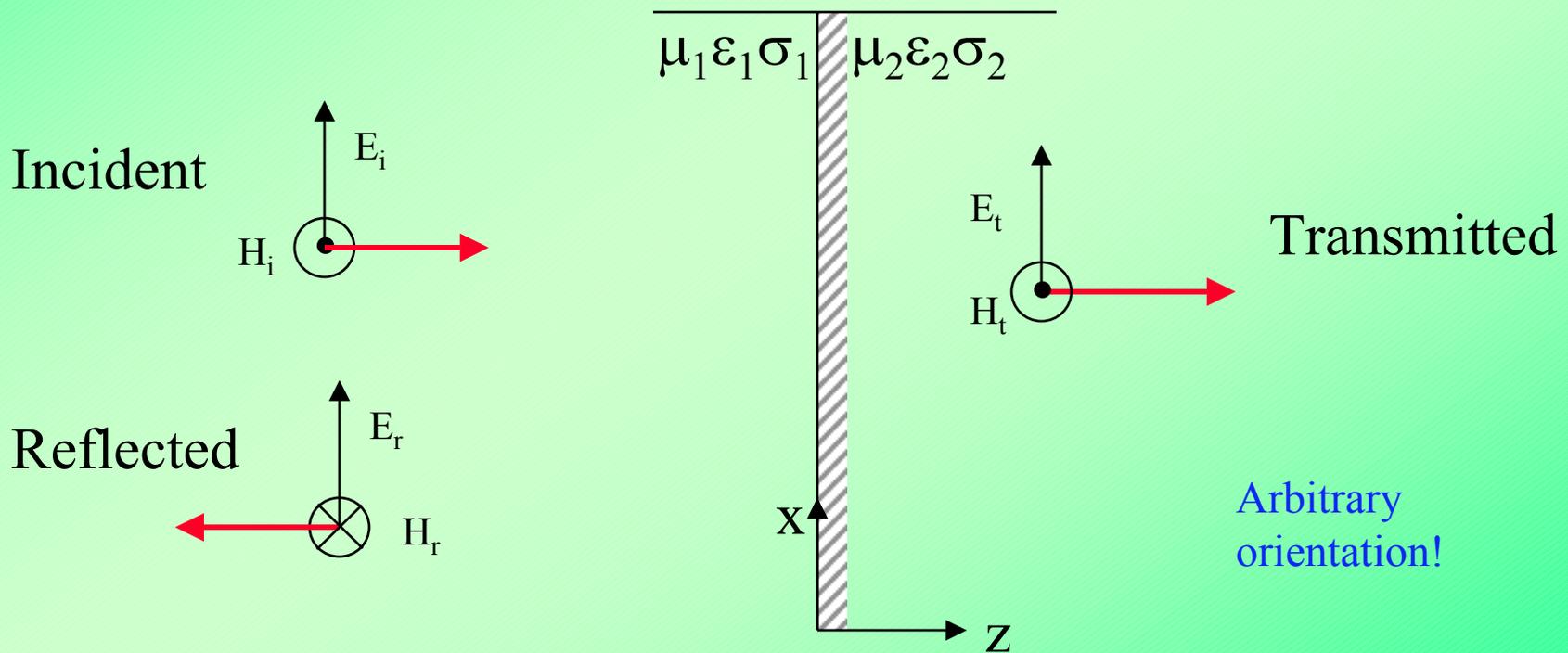
Transverse Electromagnetic (TEM) wave

Note that \mathbf{H} is also perpendicular to \mathbf{k} and also perpendicular to \mathbf{E} . This can be established from the expression for \mathbf{H} .



Plane waves at interfaces (normal incidence)

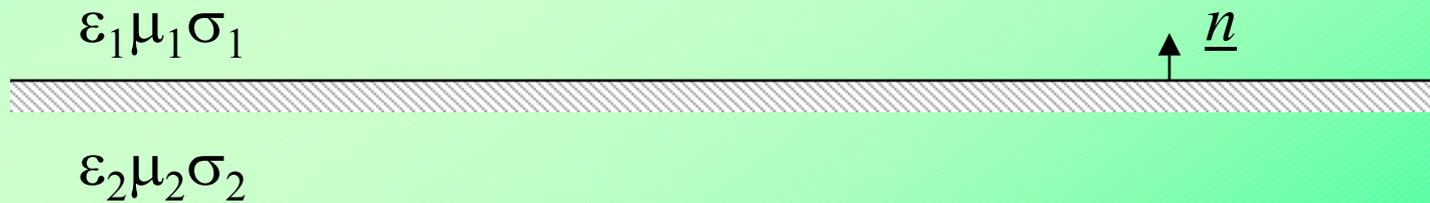
Consider a linearly polarized (in x-direction) wave travelling in the +z direction with magnitude E_i



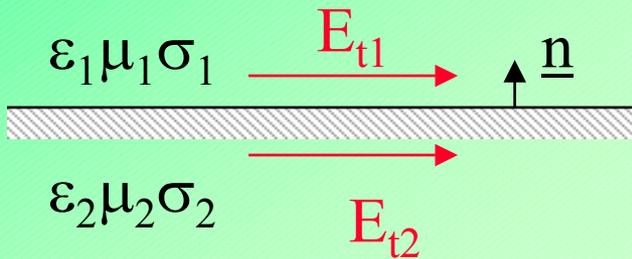
Boundary conditions

Maxwell's equations in differential form require known boundary values in order to have a complete and unique solution. The so called boundary conditions (B/C) can be derived by considering the integral form of Maxwell's equations.

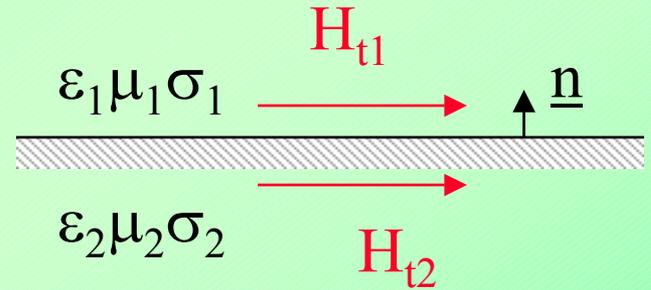
We deal with a general dielectric interface and two special cases. First the general case. For convenience we consider the boundary to be planar.



General case

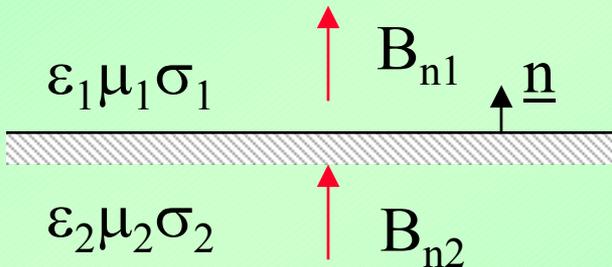


Tangential E continuous

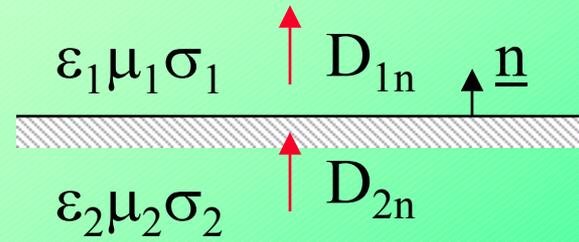


$$\underline{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

Equivalent

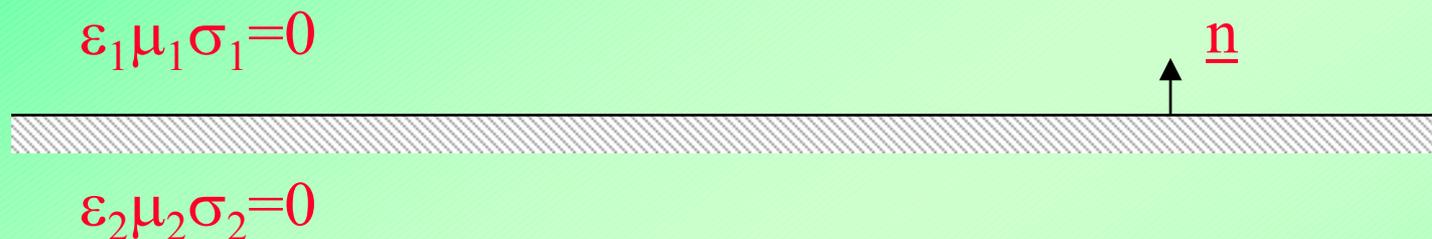


Normal B continuous



$$\underline{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

Special case (a) Lossless dielectric



$E_{t1} = E_{t2} \longrightarrow$ tangential E fields continuous)

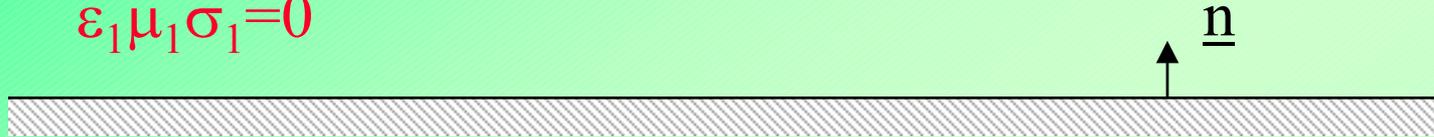
$H_{t1} = H_{t2} \longrightarrow$ tangential H fields continuous (no current)

$D_{n1} = D_{n2} \longrightarrow$ normal D fields continuous (no charge)

$B_{n1} = B_{n2} \longrightarrow$ normal B fields continuous

Special case (b) Perfect Conductor

$$\epsilon_1 \mu_1 \sigma_1 = 0$$



$$\sigma_2 \rightarrow \infty \quad \text{Perfect Electric Conductor} \quad E_{t2} = H_{t2} = 0$$

$\mathbf{E}_{t1} = 0$ \longrightarrow Tangential Electric field on conductor is zero.

$\underline{\mathbf{n}} \times \mathbf{H}_1 = \mathbf{J}_s$ \longrightarrow H field is discontinuous by the surface current

$\underline{\mathbf{n}} \cdot \mathbf{D}_1 = \rho$ \longrightarrow Normal D(E) field is discontinuous by surface charge

$\mathbf{B}_{n1} = 0$ \longrightarrow Normal B(H) field is zero on conductor.

Boundary conditions

Continuity at the boundary for the tangential fields requires:

$$E_i + E_r = E_t \quad (1)$$

$$H_i + H_r = H_t \quad (2)$$

Fix signs when defining impedance!

Now define: $\frac{E_i}{H_i} = Z_1$ $\frac{E_r}{H_r} = -Z_1$ $\frac{E_t}{H_t} = Z_2$

Substituting into (1) and (2) and eliminating E_r gives

Transmission coefficient

$$\tau = \frac{E_t}{E_i} = \frac{2Z_2}{Z_1 + Z_2}$$

Plane Wave in Dispersive Media

- Recall the Maxwell's equations:

$$\vec{\nabla} \times \vec{E} = -j\omega\vec{B}$$

$$\vec{\nabla} \times \vec{B} = j\omega\vec{D} + \vec{J}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{D} = \rho_v$$

$$\vec{E}(x, y, z; t) = \vec{E}(x, y, z)e^{j\omega t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\int \nabla \times \vec{E}(x, y, z)e^{j\omega t} = -\int \frac{\partial \vec{B}}{\partial t}$$

$$\frac{1}{j\omega} (\vec{\nabla} \times \vec{E}) = \vec{B} \Rightarrow \vec{\nabla} \times \vec{E} = -j\omega\vec{B}$$

Plane Wave in Dispersive Media

- So far, for lossless media, we considered $J=0$, and $\rho_v=0$ but, there are actually two types of current and one of them should not be ignored.
- Total current is a sum of the Source current and Conduction current.

$$\vec{J} = \vec{J}_c + \vec{J}_o$$

$$\text{set } \vec{J}_c = \sigma \vec{E}$$

$$\vec{\nabla} \times \vec{H} = j\omega \vec{D} + \vec{J}$$

$$\vec{\nabla} \times \vec{H} = j\omega \epsilon \vec{E} + \sigma \vec{E} + \vec{J}_o = j\omega \left(\epsilon - j \frac{\sigma}{\omega} \right) \vec{E} + \vec{J}_o$$

Plane Wave in Dispersive Media

Defining complex permittivity

$$\underline{\underline{\epsilon}} = \epsilon - j \frac{\sigma}{\omega}$$

Maxwell's equations in a conducting media (source free) can be written as

$$\vec{\nabla} \times \vec{E} = -j\omega\mu\vec{H}$$

$$\vec{\nabla} \times \vec{H} = j\omega\epsilon\vec{E}$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

Plane Wave in Dispersive Media

We have considered so far:

Plane Waves
in Free space

Plane Waves
in Isotropic
Dielectric

Plane Waves
in anisotropic
Dielectric

Plane Waves
in Dissipative
Media

$$\vec{\nabla} \times \vec{E} = -j\omega\mu_0\vec{H}$$

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$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon_0\vec{E}$$

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$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon\vec{E}$$

$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon\vec{E}$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

Plane Wave in Dispersive Media

Wave equation for dissipative media becomes:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -j\omega\mu\vec{\nabla} \times \vec{H}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -j\omega\mu(j\omega\epsilon\vec{E})$$

$$\vec{\nabla}^2 \vec{E} = -\omega^2\mu\epsilon\vec{E}$$

$$\vec{\nabla}^2 \vec{H} = -\omega^2\mu\epsilon\vec{H}$$

The set of plane-wave solutions are:

$$\vec{E} = \hat{x}E_0e^{-j\kappa z}$$

$$\vec{H} = \hat{y}\left(\frac{E_0}{\eta}\right)e^{-j\kappa z}$$

Plane Wave in Dispersive Media

Substituting into $\vec{\nabla}^2 \vec{E} = -\omega^2 \mu \epsilon \vec{E}$ and $\vec{\nabla}^2 \vec{H} = -\omega^2 \mu \epsilon \vec{H}$

yields the dispersion relation

$$\kappa^2 = \omega^2 \mu \epsilon$$

and

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

Is the complex intrinsic impedance of the isotropic media.

Plane Wave in Dispersive Media

Denoting the complex values:

$$\kappa = \kappa_R - j\kappa_I$$

$$\eta = |\eta| e^{j\phi}$$

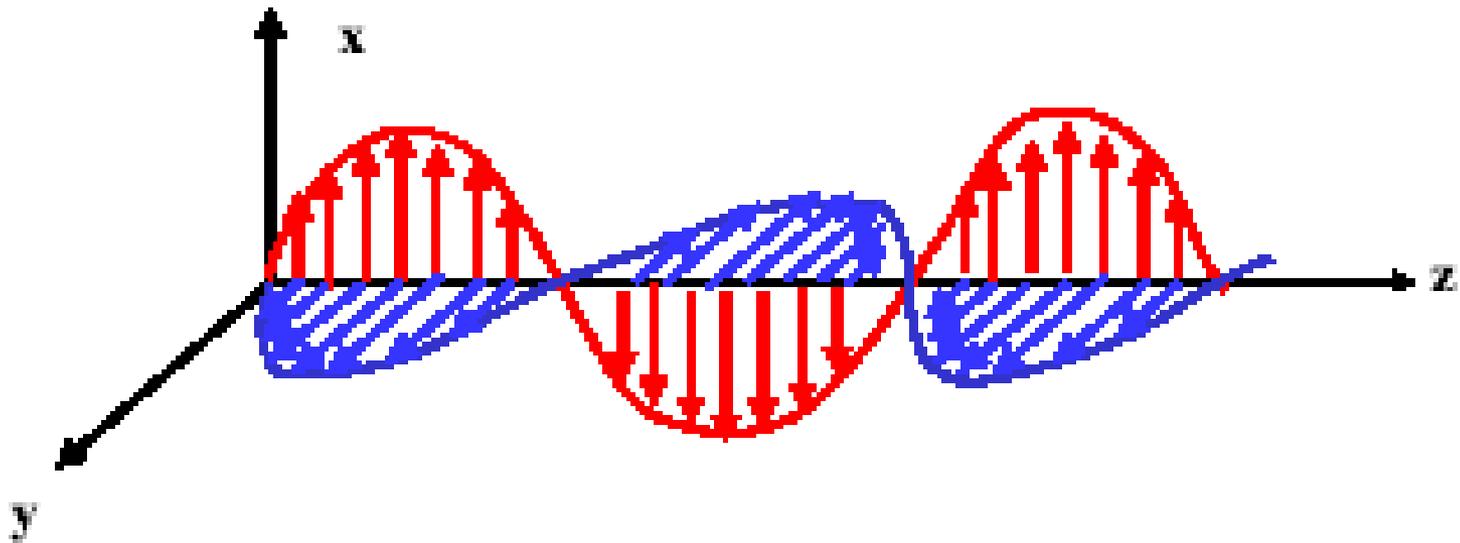
then,

$$\vec{E} = \hat{x}E_0 e^{-j\kappa z} = \hat{x}E_0 e^{-j(\kappa_R - j\kappa_I)z} = \hat{x}E_0 e^{-j\kappa_R z} e^{-\kappa_I z} = \hat{x} \underline{E_x}$$

$$\vec{H} = \hat{y} \left(\frac{E_0}{\eta} \right) e^{-j(\kappa_R - j\kappa_I)z} = \hat{y} \frac{E_0}{|\eta|} e^{-j(\kappa_R - j\kappa_I)z} e^{-j\phi}$$

Plane Wave in Dispersive Media

- Picture it: Lossless Media



Plane Wave in Dispersive Media

Loss tangent is defined from

$$\begin{aligned}\kappa &= \kappa_R - j\kappa_I = \omega\sqrt{\mu\underline{\varepsilon}} = \omega\sqrt{\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)} \\ &= \omega\sqrt{\mu\varepsilon}\sqrt{\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}\end{aligned}$$

$\frac{\sigma}{\omega\varepsilon}$ is defined as loss tangent

$$\underline{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right) = \varepsilon' - j\varepsilon''$$

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'}$$

Plane Wave in Dispersive Media

Slightly lossy case: $\frac{\sigma}{\omega\varepsilon} \ll 1$

$$\kappa = \omega\sqrt{\mu\varepsilon} \sqrt{\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} = \omega\sqrt{\mu\varepsilon} \left(1 - j\frac{\sigma}{2\omega\varepsilon}\right)$$

$$\kappa_R = \omega\sqrt{\mu\varepsilon}$$

$$\kappa_I = \omega\sqrt{\mu\varepsilon} \frac{\sigma}{2\omega\varepsilon} = \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}}$$

$$d_p = \frac{2}{\sigma} \sqrt{\frac{\varepsilon}{\mu}}$$

Plane Wave in Dispersive Media

Highly lossy case: $\frac{\sigma}{\omega\epsilon} \gg 1$

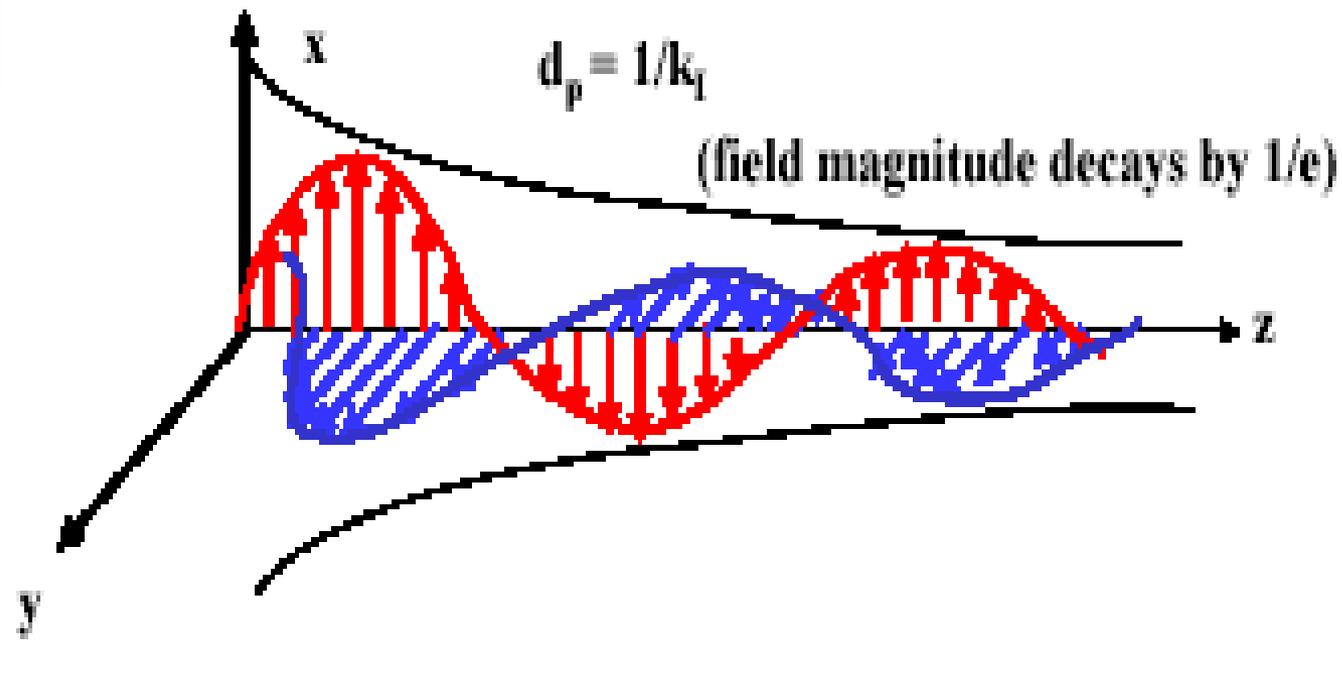
$$\kappa = \omega\sqrt{\mu\epsilon}\sqrt{\left(1 - j\frac{\sigma}{\omega\epsilon}\right)} = \omega\sqrt{\mu\epsilon}\left(\sqrt{-j\frac{\sigma}{2\omega\epsilon}}\right)$$

$$= \sqrt{\omega\mu\frac{\sigma}{2}}(1 - j)$$

$$d_p = \sqrt{\frac{2}{\omega\mu\sigma}} \approx \delta \quad \text{Skin depth}$$

Plane Wave in Dispersive Media

- Picture it: Lossy Media



Reflection & Transmission

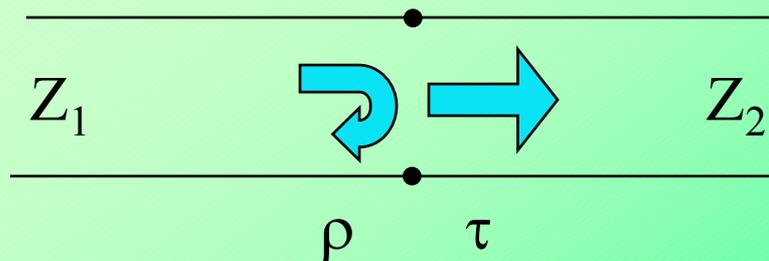
Similarly, substituting into (1) and (2) and eliminating E_t

Reflection coefficient

$$\rho = \frac{E_r}{E_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

Not $1-\rho$

We note that $\tau = 1 + \rho$, and that the values of the reflection and transmission are the same as occur in a transmission line discontinuity.



Special case (1)

(1) Medium 1: air; Medium 2: conductor

$$Z_1 = 377\Omega \gg Z_2 = Z_m = \frac{1+j}{\sigma\delta}$$

$$\text{So } E_t = \tau E_i \approx \frac{2Z_2}{Z_1} E_i$$

$$\text{then use } H_t = \frac{E_t}{Z_2} \Rightarrow H_t = \frac{2}{Z_1} E_i \approx 2H_i$$

This says that the transmitted magnetic field is **almost doubled** at the boundary before it decays according to the skin depth.

On the reflection side $H_i \approx H_r$ implying that almost all the H-field is reflected forming a standing wave.

Special case (2)

(2) Medium 1: conductor; Medium 2: air

Reversing the situation, now where the wave is incident from the conducting side, we can show that the wave is almost totally reflected within the conductor, but that the standing wave is attenuated due to the conductivity.

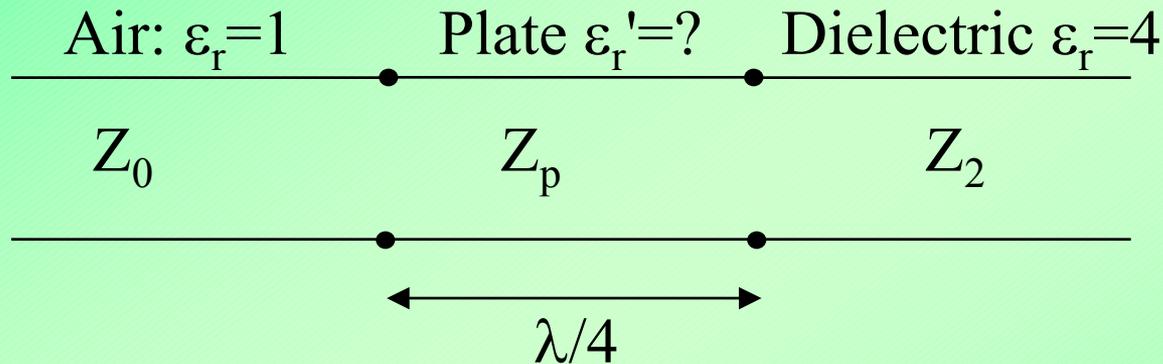
Special Case (3)

(2) Medium1: dielectric; Medium2: dielectric

$$Z_1 = \sqrt{\frac{\mu_0}{\epsilon_1}}, \quad Z_2 = \sqrt{\frac{\mu_0}{\epsilon_2}} \quad \Rightarrow \quad \rho = \frac{\sqrt{\frac{\epsilon_1}{\epsilon_2}} - 1}{\sqrt{\frac{\epsilon_1}{\epsilon_2}} + 1}$$

This result says that the reflection can be controlled by varying the ratio of the dielectric constants. The transmission analogy can thus be used for a quarter-wave matching device.

$\lambda/4$ Matching Plate



Transmission line theory tells us that for a match

$$Z_p = \sqrt{Z_0 Z_2}$$

We will see TL lectures later

$$Z_0 = 376.7\Omega, \quad Z_2 = \frac{Z_0}{\sqrt{\epsilon_r}} = \frac{376.7}{2} = 188\Omega$$

$$\text{So } Z_p = 266\Omega \quad \text{and} \quad \epsilon_r' = \frac{Z_0}{Z_2} = 2$$

Applications

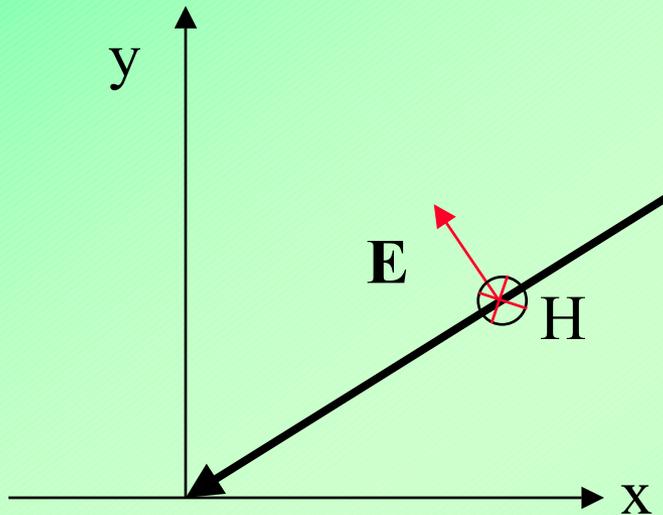
The principle of $\lambda/4$ matching is not only confined to transmission line problems! In fact, the same principle is used to eliminate reflections in **many optical devices using a $\lambda/4$ coating layer on lenses & prisms to improve light transmission efficiency.**

Similarly, a half-wave section can be used as a dielectric window. I.e. Full transparency. (Why?). In this case $Z_2=Z_0$ and the matching section is $\lambda/2$. Such devices are used to protect antennas from weather, ice snow, etc and are called **radomes**.

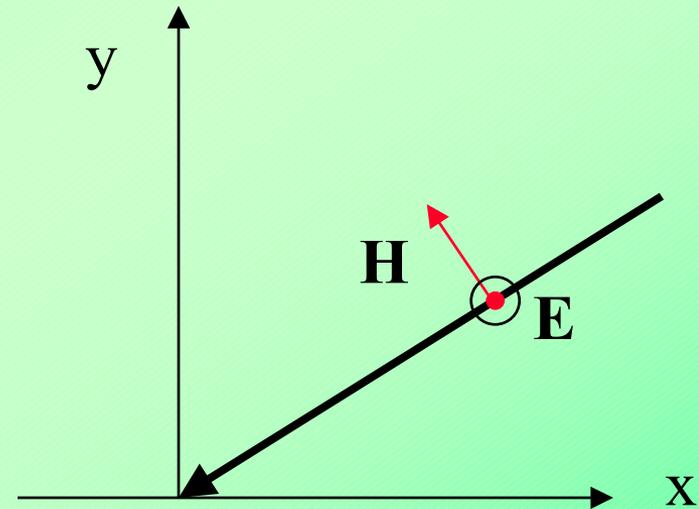
Note that both applications are frequency sensitive and that the matching section is only $\lambda/4$ or $\lambda/2$ at one frequency.

Parallel & Perpendicular Incidence

Plane of incidence is the x-y plane

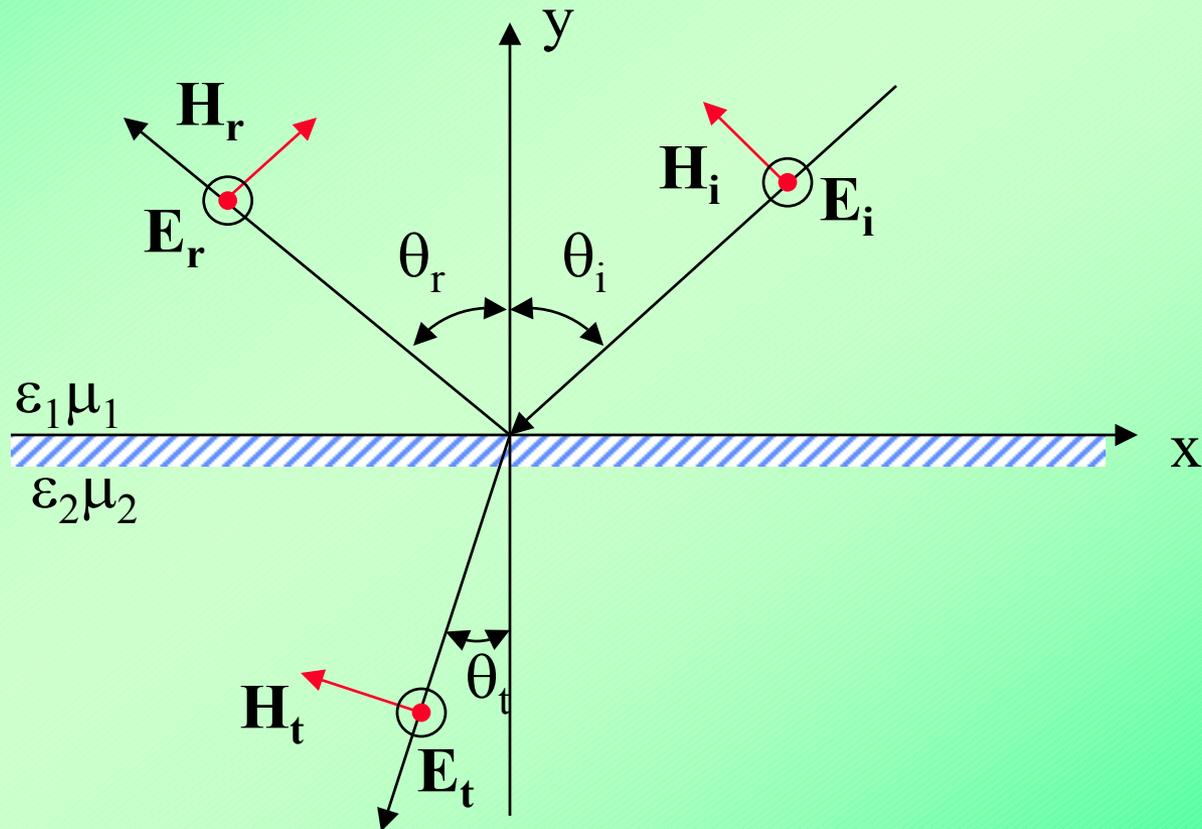


E is **Parallel** to the plane of incidence



E is **Perpendicular** to the plane of incidence

Perpendicular incidence



Transmitted field

previously

$$E_t = \hat{z} \tau_{\perp} E_0 \exp[j\beta_2 (x \sin \theta_t + y \cos \theta_t)]$$

$$= \hat{z} \tau_{\perp} E_0 \exp[j\beta_2 x \sin \theta_t] \exp[-\alpha y]$$

$\cos \theta_t = jA$

where $\alpha = \beta_2 A = \omega \sqrt{\mu_2 \epsilon_2} \sqrt{\frac{\epsilon_1}{\epsilon_2} \sin^2 \theta_i - 1}$

Physically, it is apparent that the transmitted field propagates along the surface (-x direction) but attenuates in the +y direction
This type of wave is a **surface wave** field

What about the factor $\frac{k_0}{\omega\mu_0}$?

$$\frac{k_0}{\omega\mu_0} = \frac{2\pi}{\lambda_0\omega\mu_0} = \frac{2\pi f}{c\omega\mu_0} = \frac{1}{c\mu_0} = \sqrt{\frac{\epsilon_0}{\mu_0}}$$

This term has the dimensions of admittance, in fact

$$Y_0 = \frac{1}{Z_0} = \frac{1}{\eta_0} = \sqrt{\frac{\epsilon_0}{\mu_0}}$$

where Z_0 = impedance of free space $\approx 377\Omega$

And now

$$\vec{H} = \frac{1}{\eta_0} \hat{n} \times \vec{E}$$

Propagation in conducting media

We have considered propagation in free space (perfect dielectric with $\sigma = 0$). Now consider propagation in conducting media where σ can vary from a finite value to ∞ .

Start with
$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu \frac{\partial \mathbf{J}}{\partial t} + \nabla \frac{\rho}{\epsilon}$$

Assuming no free charge and the time harmonic form, gives

$$\nabla^2 \vec{E} + \omega^2 \mu\epsilon \vec{E} = j\omega\mu\sigma \vec{E}$$

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$$

where
$$\gamma^2 = j\omega\mu\sigma - \mu\epsilon\omega^2$$

Complex propagation
coefficient due to
finite conductivity

Conduction current and displacement current

In metals, the conduction current ($\sigma\mathbf{E}$) is much larger than the displacement current ($j\omega\epsilon_0\mathbf{E}$). Only as frequencies increase to the optical region do the two become comparable.

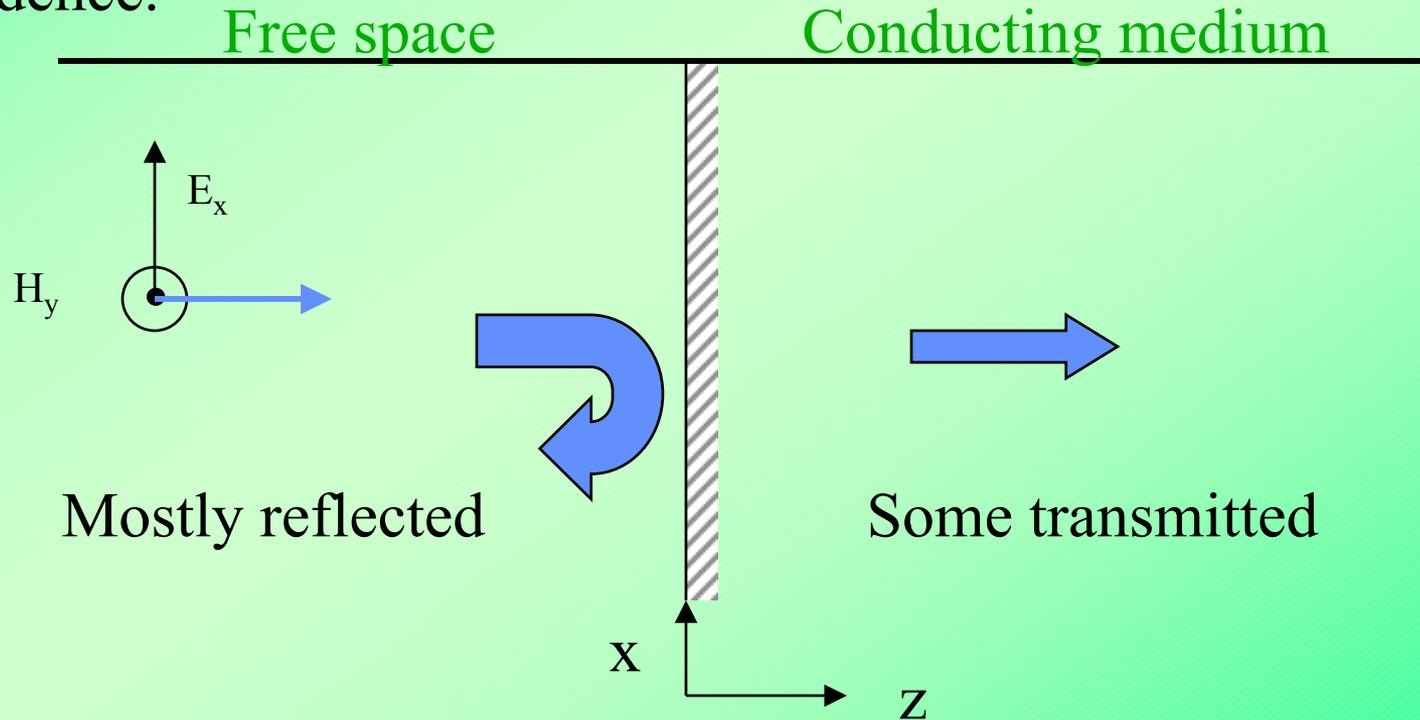
E.g. $\sigma = 5.8 \times 10^7$ for copper
 $j\omega\epsilon_0 = 2\pi \times 10^{10} \times 8.854 \times 10^{-12} = 0.556$

So retain only the $j\omega\mu\sigma$ term when considering highly conductive material at frequencies below light. The PDE becomes:

$$\nabla^2 \vec{E} - j\omega\mu_0\sigma\vec{E} = 0$$

Plane wave incident on a conductor

Consider a plane wave entering a conductive medium at normal incidence.



Mathematical solution

The equation for this is:

$$\frac{\partial^2 E_x}{\partial z^2} - j\omega\mu_0\sigma E_x = 0$$

The solution is:

$$E_x = E_0 e^{-\sqrt{j\omega\mu_0\sigma} z}$$

We can simplify the exponent:

$$\gamma = \sqrt{j\omega\mu_0\sigma} = (1+j)\sqrt{\frac{\omega\mu_0\sigma}{2}}$$

So now γ has equal real and imaginary parts.

$$E_x = E_0 e^{-\alpha z} e^{-\beta z} \quad \text{with} \quad \alpha = \beta = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

Alternatively write

$$E_x = E_0 e^{-z/\delta} e^{-jz/\delta}$$

Skin Depth

The last equation

$$E_x = E_0 e^{-z/\delta} e^{-jz/\delta}$$

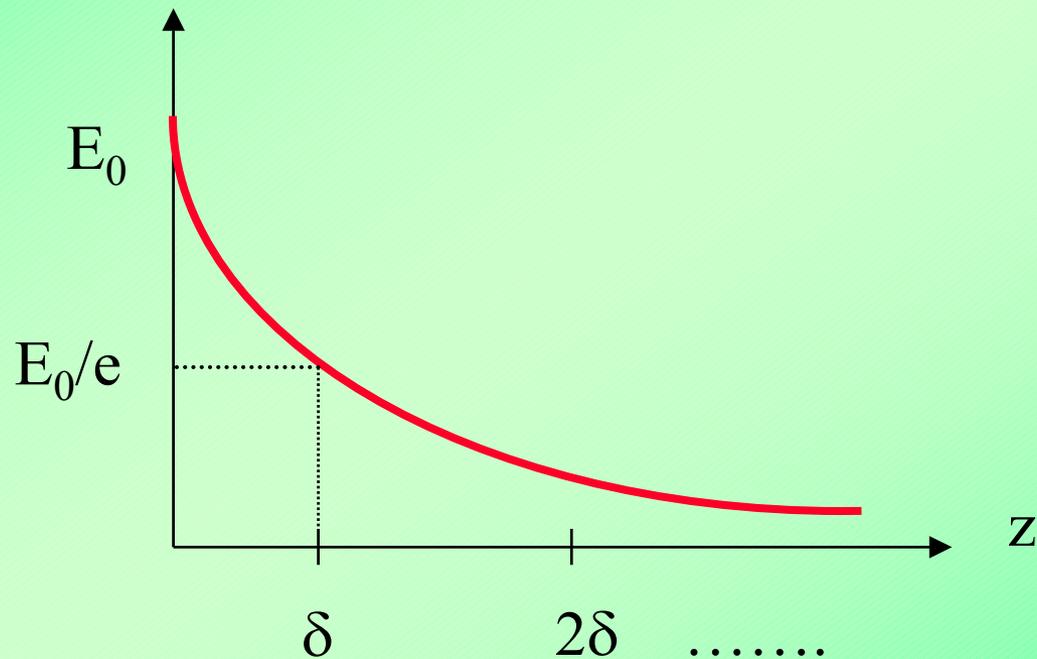
gives us the notion of skin depth:

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} = \frac{1}{\alpha} = \frac{1}{\beta}$$

On the surface at $z=0$ we have $E_x=E_0$
 at one skin depth $z=\delta$ we have $E_x=E_0/e$

field has decayed to $1/e$
 or 36.8% of value on the
 surface.

Plot of field into conductor



Examples of skin depth

Copper

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} = \frac{6.61 \times 10^{-2}}{\sqrt{f}}$$

$$\sigma = 5.8 \times 10^7 \text{ S/m}$$

at	60Hz	$\delta = 8.5 \times 10^{-3} \text{ m}$
at	1MHz	$\delta = 6.6 \times 10^{-5} \text{ m}$
at	30GHz	$\delta = 3.8 \times 10^{-7} \text{ m}$

Seawater

$$\delta = \frac{2.52 \times 10^2}{\sqrt{f}}$$

$$\sigma = 4 \text{ S/m}$$

at	1 kHz	$\delta = 7.96 \text{ m}$
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Submarine comms!

Characteristic or Intrinsic Impedance

$$Z_m$$

Define this via the material as before:

$$Z_m = \sqrt{\frac{\mu_0}{\epsilon_c}} = \sqrt{\frac{\mu_0}{\epsilon - j\frac{\sigma}{\omega}}}$$

But again, the conduction current predominates, which means the second term in the denominator is large. With this approximation we can arrive at:

$$Z_m = (1 + j) \sqrt{\frac{\omega \mu_0}{2\sigma}} = \frac{1 + j}{\sigma \delta}$$

For copper at 10GHz $Z_m = 0.026(1+j) \Omega$

Reflection from a metal surface

So a reflection coefficient at metal-air interface is

$$\rho = \frac{Z_m - Z_0}{Z_m + Z_0} \approx -1 \quad \text{since } Z_m \ll Z_0$$

We also note that as $\sigma \rightarrow \infty$, $Z_m \rightarrow 0$ and that $\rho = -1$ for the case of the perfect conductor. Thus the boundary condition for a PEC is satisfied in the limit.

The transmission coefficient into the metal is given by $\tau = 1 + \rho$

Conductors and dielectrics

Materials can behave as either a dielectric or a conductor depending on the frequency.

recall $\nabla \times H = \sigma E + j\omega \epsilon E$

Conduction current density

Displacement current density

3 choices

$\omega \epsilon \gg \sigma$ displacement current \gg conductor current \Rightarrow dielectric

$\omega \epsilon \approx \sigma$ displacement current \approx conductor current \Rightarrow quasi conductor

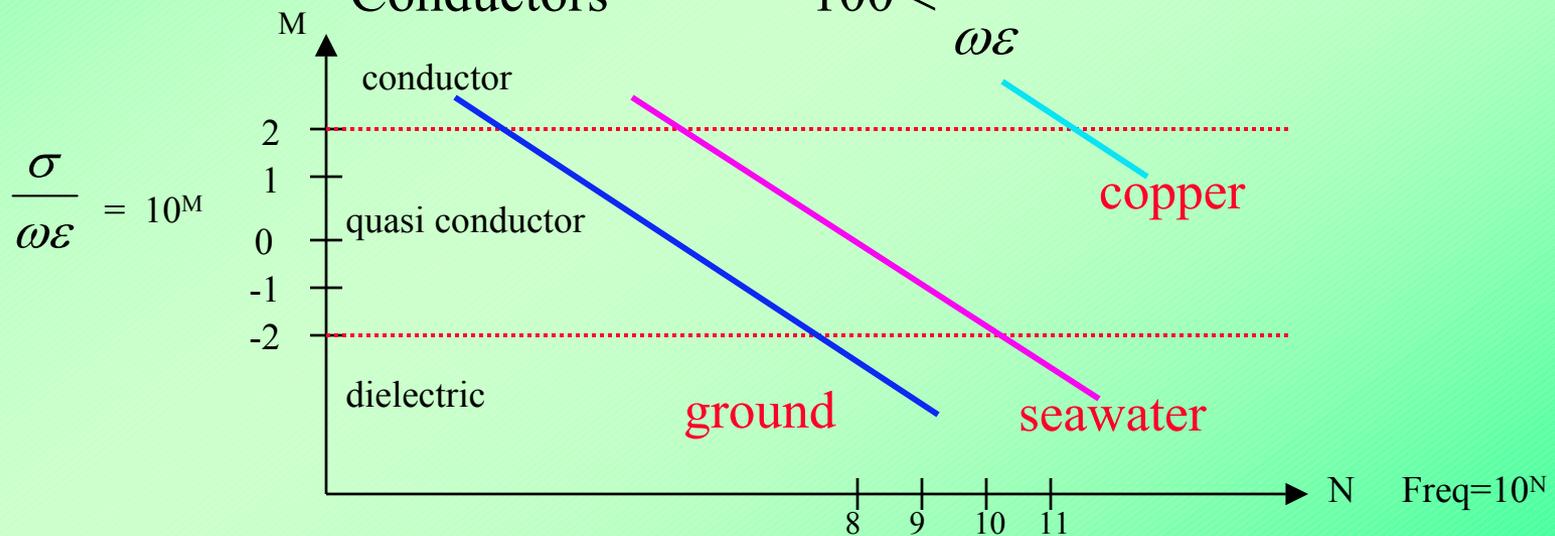
$\omega \epsilon \ll \sigma$ displacement current \ll conductor current \Rightarrow conductor

A rule for determining whether dielectric or conductor

Dielectrics $\frac{\sigma}{\omega\epsilon} < \frac{1}{100}$

Quasi Conductors $\frac{1}{100} < \frac{\sigma}{\omega\epsilon} < 100$

Conductors $100 < \frac{\sigma}{\omega\epsilon}$



General case: (both conduction & displacement currents)

From 4.4, we now retain both terms:

$$\gamma^2 = j\omega\mu\sigma - \mu\varepsilon\omega^2 = -\omega^2\mu\varepsilon\left[1 + \frac{\sigma}{j\omega\varepsilon}\right]$$

If we now let $\gamma = \alpha + j\beta$, square it and equate real and imaginary parts and then solve simultaneously for α and β . We obtain:

$$\alpha = \omega\sqrt{\mu\varepsilon} \left\{ \frac{1}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1 \right] \right\}^{\frac{1}{2}} \quad \text{Np/m}$$

$$\beta = \omega\sqrt{\mu\varepsilon} \left\{ \frac{1}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1 \right] \right\}^{\frac{1}{2}} \quad \text{rad/m}$$

Approximations

By taking a binomial expansion of the term under the radical and simplifying, we can obtain:

	<u>Good dielectric</u>	<u>Good conductor</u>
α	$\frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}$	$\sqrt{\frac{\omega\mu\sigma}{2}}$
β	$\omega\sqrt{\mu\epsilon}$	$\sqrt{\frac{\omega\mu\sigma}{2}}$
Z_w	$\sqrt{\frac{\mu}{\epsilon}}$	$\sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$

Example Problem 1:

An FM radio broadcasts signal traveling in the y -direction in air has a magnetic field given by the phasor

$$H(y) = 2.92 \times 10^{-3} e^{-j0.68\pi y} (-\hat{x} + \hat{z}j) \text{ A} \cdot \text{m}^{-1}$$

(a) Determine the frequency (in MHz) and wavelength (in m).

(b) Find the corresponding $E(y)$.

(a) we have

$$\beta = \omega \sqrt{\mu_0 \epsilon_0} = 0.68\pi \text{ rad} \cdot \text{m}^{-1}$$

from which

$$f = \frac{\omega}{2\pi} \approx 102 \text{ MHz}$$

$$\nabla \times H = \hat{x} \frac{\partial H_z}{\partial y} - \hat{z} \frac{\partial H_x}{\partial y} = j\omega \epsilon_0 E$$

$$\Rightarrow E(y) \approx 1.1 e^{-j0.68\pi y} (-\hat{x}j - \hat{z}) \text{ V} \cdot \text{m}^{-1}$$

Example Problem 2:

A uniform plane wave of frequency 10 GHz propagates in a sufficiently large sample of gallium arsenide (GaAs, $\epsilon_r \approx 12.9, \mu_r \approx 1, \tan \delta_c \approx 5 \times 10^{-5}$), which is a commonly substrate material for high-speed solid-state devices. Find (a) the attenuation constant α in np-m^{-1} , (b) phase velocity v_p in m-s^{-1} , and (c) intrinsic impedance η_c in Ω .

Since $\tan \delta_c = 5 \times 10^{-4} \ll 1$, we can use the approx for a good dielectric

(a) We have

$$\begin{aligned} \alpha &\approx \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}} = \frac{\omega \epsilon \tan \delta_c}{2} \sqrt{\frac{\mu}{\epsilon}} = \frac{2\pi \times 10^{10} \times 5 \times 10^{-4}}{2} \sqrt{\frac{\mu}{\epsilon}} \\ &= \frac{2\pi \times 10^{10} \times 5 \times 10^{-4} \sqrt{\mu_r \epsilon_r} \sqrt{\mu_0 \epsilon_0}}{2} \\ &= \frac{2\pi \times 10^{10} \times 5 \times 10^{-4}}{2 \times 3 \times 10^8} \sqrt{12.9} \approx 0.188 \text{ np-m}^{-1} \end{aligned}$$

Example Problem 2:

(b) Since phase velocity $v_p = \frac{\omega}{\beta}$

where $\beta \approx \omega\sqrt{\mu\varepsilon}$, we have

$$v_p \approx \frac{1}{\sqrt{\mu\varepsilon}} \approx \frac{3 \times 10^8}{\sqrt{12.9}} \approx 8.35 \times 10^7 \text{ m-s}^{-1}. \text{ Note that the}$$

phase velocity is ~ 3.59 times slower than in the air.

(c) The intrinsic impedance $\eta_c \approx \sqrt{\frac{\mu}{\varepsilon}} \approx \frac{377}{\sqrt{12.9}} \approx 105\Omega$.

Note that the intrinsic impedance is ~ 3.59 times smaller than that in air.

Example Problem3:

A recent survey conducted in USA indicates that ~50% of the population is exposed to average power densities of approximately $0.005 \mu\text{W}-(\text{cm})^{-2}$ due to VHF and UHF broadcast radiation. Find the corresponding amplitude of the electric and magnetic fields.

Consider the uniform plane wave propagating in a lossless medium :

$$E_x = E_0 \cos(\omega t - \beta z)$$

$$H_y = \frac{1}{\eta} E_0 \cos(\omega t - \beta z)$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ and $\eta = \sqrt{\frac{\mu}{\varepsilon}}$. The Poynting vector for this wave is given by

$$\bar{P} = \bar{E} \times \bar{H} = \hat{z} E_0 \left(\frac{E_0}{\eta} \right) \cos^2(\omega t - \beta z) = \hat{z} \frac{E_0^2}{2\eta} [1 + \cos 2(\omega t - \beta z)]$$

Example Problem3:

$$S_{av} = \frac{1}{T_p} \int_0^{T_p} \bar{P}(z, t) dt = \frac{1}{T_p} \int_0^{T_p} \hat{z} \frac{E_0^2}{2\eta} [1 + \cos 2(\omega t - \beta z)] dt$$

$$\Rightarrow S_{av} = \hat{z} \frac{E_0^2}{2\eta}$$

$$S_{av} = \hat{z} \frac{E_0^2}{2\eta} = 0.005 \mu W - (cm)^{-2}$$

$$\text{so } E_0 \approx \sqrt{2 \times 377 \times 5 \times 10^{-9} / 10^{-4}} \approx 194 mV - m^{-1}$$

$$H_0 = \frac{E_0}{\eta} = \frac{194 mV - m^{-1}}{377 \Omega} = 515 \mu A - m^{-1}$$