RF and Microwave Physics

Solutions to Homework Set#1

Problem 1.

$i. E + F$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{3\times 3} + \underbrace{\begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \\ 1/2 & -1/2 & 1/2 \end{pmatrix}}_{3\times 3} = \underbrace{\begin{pmatrix} 3/2 & 1/2 & 1/2 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}}_{3\times 3}$$

$ii. E - F$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{3\times 3} - \underbrace{\begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \\ 1/2 & -1/2 & 1/2 \end{pmatrix}}_{3\times 3} = \underbrace{\begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1 & 1 & -2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix}}_{3\times 3}$$

$iii. 4B - 2H$: *Does not exist*

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$iv.\ -3(C+2A)$

$$\begin{pmatrix} -3 & -30 \\ -3 & -63 \end{pmatrix}$$

$v.\ HE+7H$

$$\underbrace{\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}}_{2\times 3}\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{3\times 3}+7\underbrace{\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}}_{2\times 3}=\underbrace{\begin{pmatrix} 12 & 34 & 11 \\ 25 & 13 & 37 \end{pmatrix}}_{2\times 3}$$

$vi.\ GG^T$

$$\underbrace{\begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}}_{3\times 2}\overbrace{\underbrace{\begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}}_{2\times 3}}^{G^T}=\underbrace{\begin{pmatrix} 9 & -3 & 3 \\ -3 & 5 & 1 \\ 3 & 1 & 2 \end{pmatrix}}_{3\times 3}$$

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Problem1 b.

i. $CD = DC$

$$\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$$

ii. $EF = FE$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \\ 1/2 & -1/2 & 1/2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} = FE$$

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Problem 2.

(i)

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

(ii)

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \Rightarrow \text{first compute } A^2$$

$$A^2 = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \Rightarrow A^n = I_2 \text{ for n even}$$

$$A^3 = A^2 \cdot A = I_2 \cdot A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} = A$$

$A^n = A$ for n odd

Problem 2.

$$iii.\ A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

De Moivre's Theorem: $[\cos\theta + j\sin\theta]^n = \left[e^{j\theta}\right]^n = \cos(n\theta) + j\sin(n\theta)$

so $(\cos n\theta + j\sin n\theta)(\cos\theta + j\sin\theta) = \cos(n+1)\theta + j\sin(n+1)\theta$

$$= (\cos n\theta\cos\theta - \sin n\theta\sin\theta) + j(\cos n\theta\sin\theta + \cos\theta\sin n\theta)$$

Thus equation of the real and imaginary parts gives us some general trig identities . Assume inductivel y that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

(the basis at $n = 1$ is free), and then

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{n+1} = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos n\theta\cos\theta - \sin n\theta\sin\theta & \cos n\theta\sin n\theta + \cos\theta\sin n\theta \\ -\cos n\theta\sin\theta - \cos\theta\sin n\theta & \cos n\theta\cos\theta - \sin n\theta\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(n+1)\theta & \sin(n+1)\theta \\ -\sin(n+1)\theta & \cos(n+1)\theta \end{pmatrix}$$

Problem 3.

We proceed by row operation and factoring :

$$\begin{vmatrix} x & -6 & -1 \\ 3 & -2x & x-4 \\ -2 & 3x & x-2 \end{vmatrix} \overset{(R_1-R_2)}{=} \begin{vmatrix} x-3 & -6+2x & 3-x \\ 3 & -2x & x-4 \\ -2 & 3x & x-2 \end{vmatrix} = (x-3)\begin{vmatrix} 1 & 2 & -1 \\ 3 & -2x & x-4 \\ -2 & 3x & x-2 \end{vmatrix}$$

(using $C_2 \leftarrow C_2 - 2C_1, C_3 \leftarrow C_3 + C_1$) $= (x-3)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2x-6 & x-1 \\ -2 & 3x+4 & x-4 \end{vmatrix}$

$$= (x-3)\begin{vmatrix} -2x-6 & x-1 \\ 3x+4 & x-4 \end{vmatrix} = (x-3)\left(-5x^2 + x + 28\right)$$

Thus the solutions are $x = 3, \dfrac{1 \pm \sqrt{561}}{10}$

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Problem 4.

$$f(x) = x^{3/2} \quad 0 \le x \le 1$$

$$V = \pi \int_0^1 \left(x^{3/2}\right)^2 dx = \pi \int_0^1 x^3 dx = \frac{\pi}{4}$$

$$f(x) = sin\ x \quad 0 \le x \le \pi/2$$

$$V = \pi \int_0^{\pi/2} sin^2\ x\ dx = \frac{\pi}{2}\left[x - cos\ xsix\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$$

$$f(x) = \frac{1}{\sqrt{\pi}}\frac{1}{x} \quad 1 \le x \le \infty$$

$$V = lim_{p\to\infty} \int_1^p \pi \frac{dx}{\pi x^2} = lim_{p\to\infty}\left[-\frac{1}{x}\right]_1^p = lim_{p\to\infty}\left(1 - \frac{1}{p}\right) = 1$$

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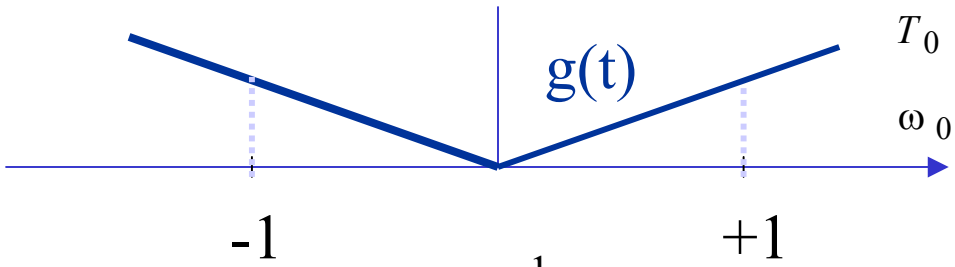
Problem 5.



$$T_0 = 2$$

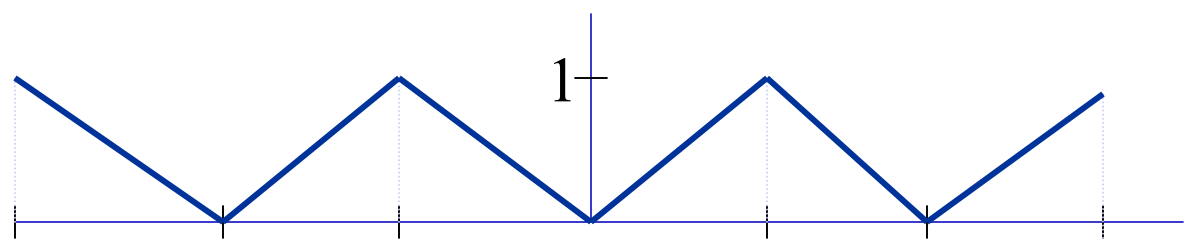
$$\omega_0 = \frac{2\pi}{T_0} = \pi$$

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)dt = \int_0^1 tdt = \left.\frac{t^2}{2}\right|_0^1 = 1/2$$

$b_0 = 0$ since $g(t)$ is even.

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos n\omega_o t dt = 2\int_0^1 t\cos n\pi t dt = \frac{2}{n^2\pi^2}\left[\cos(n\pi)-1\right] = \begin{cases} 0 & \text{, n even} \\ -\left(\frac{2}{n\pi}\right)^2 & \text{, n odd} \end{cases}$$

$$\Rightarrow \varphi(t) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t)$$



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Problem 6.

$$\kappa^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2} \qquad 0 < x < 1,\ t > 0$$

$$\theta(0,t) = 0, \qquad \frac{\partial \theta}{\partial x}(1,t) = 0$$

$$\frac{\partial \theta}{\partial t}(x,0) = 0$$

$$\theta(x,t) = \mathrm{X}(x)T(t) \Rightarrow \kappa^2 \frac{x''}{x} = \frac{\ddot{T}}{T} \Rightarrow \frac{x''}{x} = \frac{1}{\kappa^2}\frac{\ddot{T}}{T} = \lambda$$

$$\Rightarrow \begin{cases} x'' = \lambda x \\ \ddot{T} = \kappa^2 \lambda T \end{cases}$$

Problem 6.

$$BC: \quad \theta(0,t) = 0 \Rightarrow X(0) = 0$$

$$\frac{\partial\theta}{\partial x}(1,t) = 0 \Rightarrow X''(1) = 0$$

$$IC: \quad \theta(x,0) = X$$

$$\frac{\partial\theta}{\partial t}(x,0) = 0 \Rightarrow \dot{T}(0) = 0$$

$$X'' = \lambda X \qquad X(0) = 0, X'(1) = 0$$

Case 1: $\lambda = \beta^2 > 0$

$$X = A \sinh \beta x + B \cosh \beta x$$

$$X' = A\beta \cosh \beta x + B\beta \sinh \beta x$$

Problem 6.

$$\left.\begin{aligned} X(0) = 0 &\Rightarrow B = 0 \\ X'(1) = 0 &\Rightarrow A = 0 \end{aligned}\right\}$$

$\Rightarrow X(0) = 0$ is the only solution for this case since $\beta \neq 0$

Case 2 : $\lambda = -\beta^2 < 0$

$$X = A\,sin\,\beta x + B\,cos\,\beta x$$

$$X(0) = 0 \Rightarrow B = 0$$

$$X' = A\beta\,cos\,\beta x$$

$$X'(1) = 0 \Rightarrow A\beta\,cos\,\beta = 0 \Rightarrow A = 0 \Rightarrow X(x) = 0$$

or $cos\,\beta = 0$ (since $sin\,\beta \neq 0$)

Problem 6.

$$\beta = \frac{(2n-1)\pi}{2} \quad n = 1,2,3,\ldots$$

$$X(x) = A\sin\left(\frac{(2n-1)\pi}{2}x\right), \lambda = \frac{-(2n-1)^2\pi^2}{4}, \; n = 1,2,\ldots$$

Case 3 : $\lambda = 0$

$$X = Ax + B$$

$$\left.\begin{array}{l} X(0) = 0 \Rightarrow B = 0 \\ X'(1) = 0 \Rightarrow A = 0 \end{array}\right\} \Rightarrow X(x) = 0 \text{ is the only solution.}$$

Summarizing :

$$X_n = A_n \sin\left(\frac{(2n-1)\pi}{2}x\right), \lambda = \frac{-(2n-1)^2\pi^2}{4}, \; n = 1,2,\ldots$$

Problem 6.

$$\ddot{T}_n = \kappa^2 \lambda_n T_n = -\kappa^2 \left(\frac{-(2n-1)^2 \pi^2}{4} \right) T_n$$

$$T_n = C_n \, sin\left(\frac{\kappa(2n-1)\pi}{2} t \right) + d_n \, cos\left(\frac{\kappa(2n-1)\pi}{2} t \right)$$

Using $\dot{T}_n(0) = 0 \Rightarrow C_n = 0$

$$T_n = d_n \, cos\left(\frac{\kappa(2n-1)\pi}{2} t \right)$$

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Problem 6.

$$\theta_n(x,t) = X(x)T(t) = A_n \sin\left(\frac{(2n-1)\pi}{2}x\right) d_n \cos\left(\frac{\kappa(2n-1)\pi}{2}t\right)$$

$$= p_n \sin\left(\frac{(2n-1)\pi}{2}x\right) \cos\left(\frac{\kappa(2n-1)\pi}{2}t\right)$$

with $p_n = A_n d_n$

$$\Rightarrow \theta(x,t) = \sum_{n=1}^{\infty} p_n \sin\left(\frac{(2n-1)\pi}{2}x\right) \cos\left(\frac{\kappa(2n-1)\pi}{2}t\right)$$

Apply any unseparated condition : $\theta(x,0) = x,$

$$\Rightarrow X(x,0) = x \Rightarrow X(x) = \sum_{n=1}^{\infty} p_n \sin\left(\frac{(2n-1)\pi}{2}x\right)$$

Hence $f(x) = x$ and $F_n(x) = \sin\left(\frac{(2n-1)\pi}{2}x\right)$

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Problem 6.

$$\therefore\ p_n = \frac{\int_0^1 x\,sin\left(\frac{(2n-1)\pi}{2}x\right)dx}{\int_0^1 sin^2\left(\frac{(2n-1)\pi}{2}x\right)dx}$$

We applied the orthogonality condition $\int_0^1 F_n(x)F_m(x)dx = 0$ for $n \neq m$

$$p_n = \frac{\left[\frac{4}{(2n-1)^2\pi^2}sin\left(\frac{(2n-1)\pi}{2}x\right) - \frac{2x}{(2n-1)\ \pi}cos\left(\frac{(2n-1)\pi}{2}x\right)\right]_0^1}{\left[\frac{1}{2}x - \frac{1}{2}\left(\frac{2}{(2n-1)\pi}\right)cos\left(\frac{(2n-1)\pi}{2}x\right)sin\left(\frac{(2n-1)\pi}{2}x\right)\right]_0^1}$$

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Problem 6.

$$p_n = \frac{8}{(2n-1)^2\pi^2} sin\left(\frac{(2n-1)\pi}{2}\right) \quad \left[sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n+1}\right]$$

$$\therefore \theta(x,t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^2\pi^2} sin\left(\frac{(2n-1)\pi}{2}\right) sin\left(\frac{(2n-1)\pi}{2}x\right) cos\left(\frac{\kappa(2n-1)\pi}{2}t\right)$$