Problem Set 4 Thursday June 19, 2003

Problem 1:

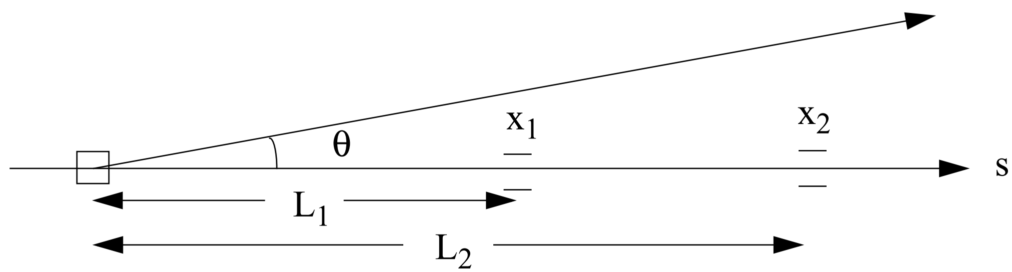

Figure for problem1

1a) The figure shows a simple transport line consisting of a corrector followed by two bpms. The corrector is separated from the bpms by drift spaces L_1 and L_2. Determine the response matrix for this corrector/bpm configuration.

Answer: $\mathbf{x_1 = L_1\theta}$ **,** $\mathbf{x_2 = L_2\theta}$ **,** $R = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$

1b) Determine the pseudoinverse of the response matrix R in 9a.

Answer: $R^{-1} = (R^T R)^{-1} R^T = \left[\begin{bmatrix} L_1 & L_2 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \right]^{-1} \begin{bmatrix} L_1 & L_2 \end{bmatrix} = \left(\frac{1}{L_1^2 + L_2^2} \right) \begin{bmatrix} L_1 & L_2 \end{bmatrix}$

1c) Derive a formula for the corrector kick angle change that minimizes the displacements at the two bpms.

Answer: $\begin{bmatrix}\theta\end{bmatrix} = R^{-1}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\dfrac{L_1x_1 + L_2x_2}{L_1^2 + L_2^2}\end{bmatrix}$

1d) Determine the SVD of the response matrix R = $U\,S\,V^T$. The eigenvalues of RR^T (R^TR) are the squares of the singular values and the normalized eigenvectors of RR^T and R^TR are the columns of U and V respectively.

Answer: $RR^T = \begin{bmatrix}L_1\\L_2\end{bmatrix}\begin{bmatrix}L_1 & L_2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}L_1^2 & L_1L_2\\ L_1L_2 & L_2^2\end{bmatrix}\end{bmatrix}$, $det(RR^T - \lambda I) = (L_1^2 - \lambda)(L_2^2 - \lambda) - (L_1L_2)^2 = 0$

$\lambda_0 = 0, \lambda_1 = (L_1^2 + L_2^2)$

Find the normalized eigenvectors of RR^T to get the columns of U.

$u_{\lambda_1 = (L_1^2 + L_2^2)} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$; $\begin{bmatrix}-L_2^2 & L_1L_2\\ L_1L_2 & -L_1^2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}-L_2^2u_1 + L_1L_2u_2\\ L_1L_2u_1 - L_1^2u_2\end{bmatrix}$; $u_{\lambda_1 = (L_1^2 + L_2^2)} = \left(\dfrac{1}{\sqrt{L_1^2 + L_2^2}}\right)\begin{bmatrix}L_1\\L_2\end{bmatrix}$

$u_{\lambda_0 = 0} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$; $\begin{bmatrix}L_1^2 & L_1L_2\\ L_1L_2 & L_2^2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}L_1^2u_1 + L_1L_2u_2\\ L_1L_2u_1 + L_2^2u_2\end{bmatrix}$; $u_{\lambda_0 = 0} = \left(\dfrac{1}{\sqrt{L_1^2 + L_2^2}}\right)\begin{bmatrix}L_2\\-L_1\end{bmatrix}$

$$R^T R = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} L_1^2 + L_2^2 \end{bmatrix}$$

$$det(R^T R - \lambda_1 I) = L_1^2 + L_2^2 - \lambda_1 = 0$$

The eigenvalue for $R^T R$ is repeated. The normalized eigenvector = V = V^T = [1] = $v_{\lambda 1}$ = [v_1].

So the final answer is easily verified by multiplication.

$$R = USV^T = \begin{bmatrix} u_{\lambda_1 = (L_1^2 + L_2^2)} & u_{\lambda_0 = 0} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} \\ 0 \end{bmatrix} \begin{bmatrix} v_1 \end{bmatrix} = \left(\frac{1}{\sqrt{L_1^2 + L_2^2}} \right) \begin{bmatrix} L_1 & L_2 \\ L_2 & -L_1 \end{bmatrix} \begin{bmatrix} \sqrt{L_1^2 + L_2^2} \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$R = \begin{bmatrix} L_1 & L_2 \\ L_2 & -L_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

1e) Determine the value of the corrector angle that minimizes the position of the beam at the bpms by minimizing the function:

$\chi^2 = (x_1 - L_1\theta)^2 + (x_2 - L_2\theta)^2$.

Answer:

$$\frac{\partial \chi^2}{\partial \theta} = 0 = -2L_1(x_1 - L_1\theta) - 2L_2(x_2 - L_2\theta) \text{ ; } \theta = \frac{L_1 x_1 + L_2 x_2}{L_1^2 + L_2^2}$$

Problem 2:

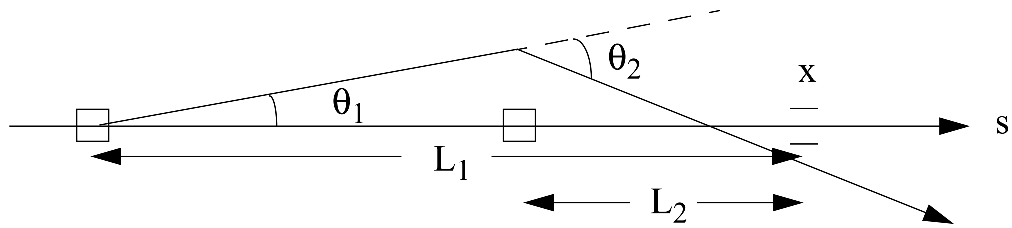

Figure for problem 2

2a) The figure shows a simple transport line consisting of two correctors followed by one bpm. The corrector is separated from the bpms by drift spaces L_1 and L_2. Determine the response matrix for this corrector/bpm configuration.

Answer: $\mathbf{x = L_1\theta_1}$, $\mathbf{x = L_2\theta_2}$, $x = L_1\theta_1 + L_2\theta_2$ **,** $R = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$

2b) Determine the pseudoinverse of the response matrix R in 2a.

Answer: There is no pseudo inverse as it stands because the matrix $\mathbf{R^TR}$ has determinant 0! What to do, SVD to the rescue!

$$R^{-1} = (R^T R)^{-1} R^T = \left[\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} L_1 & L_2 \end{bmatrix}\right]^{-1} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} L_1^2 & L_1 L_2 \\ L_1 L_2 & L_2^2 \end{bmatrix}^{-1} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

2c) Determine the SVD of the response matrix R = $U\,S\,V^T$. The eigenvalues of RR^T (R^TR) are the squares of the singular values and the normalized eigenvectors of RR^T and R^TR are the columns of U and V respectively.

Answer: $RR^T = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} L_1^2 + L_2^2 \end{bmatrix}$ **,** $\quad det(RR^T - \lambda I) = \lambda - (L_1^2 + L_2^2) = 0$

$\lambda_0 = (L_1^2 + L_2^2)$

Find the normalized eigenvectors of $\mathbf{RR^T}$ to get the columns of U. The normalized eigenvector = U = $\mathbf{U^T}$ = [1] = $\mathbf{u_{\lambda 1}}$ = [$\mathbf{u_1}$].

Find the normalized eigenvectors for the matrix $\mathbf{R^TR}$. The normalized eigenvectors are the columns of V.

$$R^T R = \begin{bmatrix} L_1^2 & L_1 L_2 \\ L_1 L_2 & L_2^2 \end{bmatrix}$$

But, this matrix is the same as $\mathbf{RR^T}$ from problem 9! with the same eigenvalues. So, we can immediately write down the matrix V and $\mathbf{V^T}$.

$$V = \frac{1}{\sqrt{L_1^2 + L_2^2}} \begin{bmatrix} L_1 & L_2 \\ L_2 & -L_1 \end{bmatrix} = V^T = V^{-1}$$

So the final answer is. $R = USV^T = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} \sqrt{L_1^2 + L_2^2} & 0 \end{bmatrix} \left(\frac{1}{\sqrt{L_1^2 + L_2^2}} \begin{bmatrix} L_1 & L_2 \\ L_2 & -L_1 \end{bmatrix} \right) = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$

2d) Determine the pseudoinverse of the matrix R using the SVD result in 2c.

Answer:

$$R = VS^{-1}U^{T} = \frac{1}{\sqrt{L_1^2 + L_2^2}}\begin{bmatrix} L_1 & L_2 \\ L_2 & -L_1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{L_1^2 + L_2^2}} \\ 0 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix} = \left(\frac{1}{L_1^2 + L_2^2}\right)\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

2e) Use the result of 2d to determine the corrector kick angles that minimize the position at the bpm.

Answer:

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = R^{-1} x = \frac{1}{L_1^2 + L_2^2}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \frac{1}{L_1^2 + L_2^2}\begin{bmatrix} L_1 x \\ L_2 x \end{bmatrix} = \frac{x}{L_1^2 + L_2^2}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

In equation form the kick angles are:

$\theta_1 = \mathbf{L_1 x / (L_1^2 + L_2^2)}$, $\theta_2 = \mathbf{L_2 x / (L_1^2 + L_2^2)}$

This is the common mode solution (angles have the same sign as x) and is stable. The differential mode (angles have opposite signs) is obviously unstable.

2f) The method of Lagrange multipliers is used to minimize a function subject to a constraint. In this problem, the constraint is that the difference between the bpm position must always equal the sum of the two corrector kicks. The constraint can be expressed by the function:

$G(\theta_1,\theta_2) = x - L_1\theta_1 - L_2\theta_2 = 0$

In this problem, the function (χ^2) to minimize with this constraint is the sum of the squares of the corrector kick angles: $\chi^2 = \theta_1{}^2 + \theta_2{}^2$.

Given the function:

$F(\theta_1,\theta_2,\lambda) = \chi^2 + \lambda G(\theta_1,\theta_2)$,

Derive formulas for the corrector kick angles by minimizing $F(\theta_1,\theta_2,\lambda)$ with respect to the angles and λ.

Answer: $\frac{\partial F}{\partial \theta_1} = 2\theta_1 + \lambda L_1 = 0$ **;** $\frac{\partial F}{\partial \theta_2} = 2\theta_2 + \lambda L_2 = 0$ **;** $\frac{\partial F}{\partial \lambda} = x - L_1\theta_1 - L_2\theta_2 = 0$

From the first two equations:

$\theta_2 = \left(\frac{L_2}{L_1}\right)\theta_1$ **; which is what you would expect θ_2 should be < θ_1 for a well behaved solution. Finally, the result is the same as for SVD.**

$\theta_1 = \frac{L_1 x}{L_1^2 + L_2^2}$ **;** $\theta_2 = \frac{L_2 x}{L_1^2 + L_2^2}$ **;** $\lambda = \frac{-2x}{L_1^2 + L_2^2}$

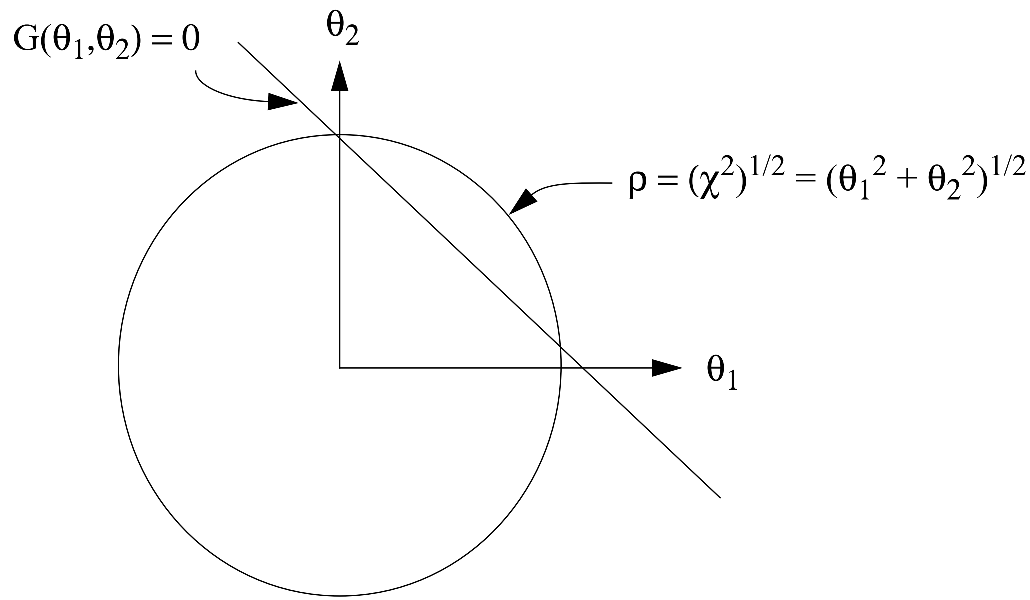

Figure for problem 2g

2g) To illustrate the least squares minimization obtained from the SVD/pseudoinverse method and the method of Lagrange multipliers, consider the constraint function $G(\theta_1,\theta_2)$ and χ^2 in the θ_1, θ_2 plane: Show that the values for θ_1 and θ_2 given by both methods are simply the point where the circle defined by ρ is tangent to the line defined by $G(\theta_1,\theta_2) = 0$.

Answer: Here is a possible solution noting that a circle intersects a line in 2, 1 or 0 points. The condition for a single point is when the line is tangent to the circle. For this solution, simply substituteθ_2 from the function $G(\theta_1,\theta_2)$ into the function for ρ and solve for θ_1 in terms of ρ.

$\theta_2 = (\mathbf{x - L_1}\theta_1) / \ \mathbf{L_2}$; $\rho^2 = \theta_1^{\mathbf{2}} + \theta_2^{\mathbf{2}}$

$\rho^2 = \theta_1^{\mathbf{2}} + (\mathbf{x - L_1}\theta_1)^2 / \mathbf{L_2^2} = \theta_1^{\mathbf{2}}(\mathbf{1 + L_1}^2 / \ \mathbf{L_2^2}) + \mathbf{2xL_1}\theta_1 / \mathbf{L_2^2 + x^2 / L_2^2}$ **;**

$\theta_1^{\mathbf{2}} + (\mathbf{x - L_1}\theta_1)^2 / \mathbf{L_2^2} = \theta_1^{\mathbf{2}}(\mathbf{1 + L_1}^2 / \ \mathbf{L_2^2}) - \mathbf{2xL_1}\theta_1 / \mathbf{L_2^2 + x^2 / L_2^2} - \rho^2 = 0$

$$\theta_1 = \frac{\dfrac{2xL_1}{L_2^2} \pm \sqrt{\dfrac{4x^2L_1^2}{L_2^4} - 4\left(\dfrac{x^2}{L_2^2} - \rho^2\right)\left(1 + \dfrac{L_1^2}{L_2^2}\right)}}{2\left(1 + \dfrac{L_1^2}{L_2^2}\right)} = \frac{xL_1 \pm \sqrt{x^2L_1^2 - (x^2 - L_2^2\rho^2)(L_1^2 + L_2^2)}}{L_1^2 + L_2^2}$$

Solve this quadratic equation for θ_1**. There are two solutions only when the term in the square root is > 0. There is only a single solution is when the term in the square root equals 0. This is the same solution as given by SVD/pseudoinverse and method of Lagrange multipliers.**

$$\theta_2 = -\frac{L_1}{L_2}\theta_1 + \frac{x}{L_2} = -\frac{L_1}{L_2}\left(\frac{xL_1}{L_1^2 + L_2^2}\right) + \frac{x}{L_2} = \frac{xL_2}{L_1^2 + L_2^2}$$

Finally, the value of ρ^2 **for these values is:** $\rho^2 = \dfrac{x^2}{L_1^2 + L_2^2}$ **For this value of** ρ^2 **the square root of the above expression equals 0.**