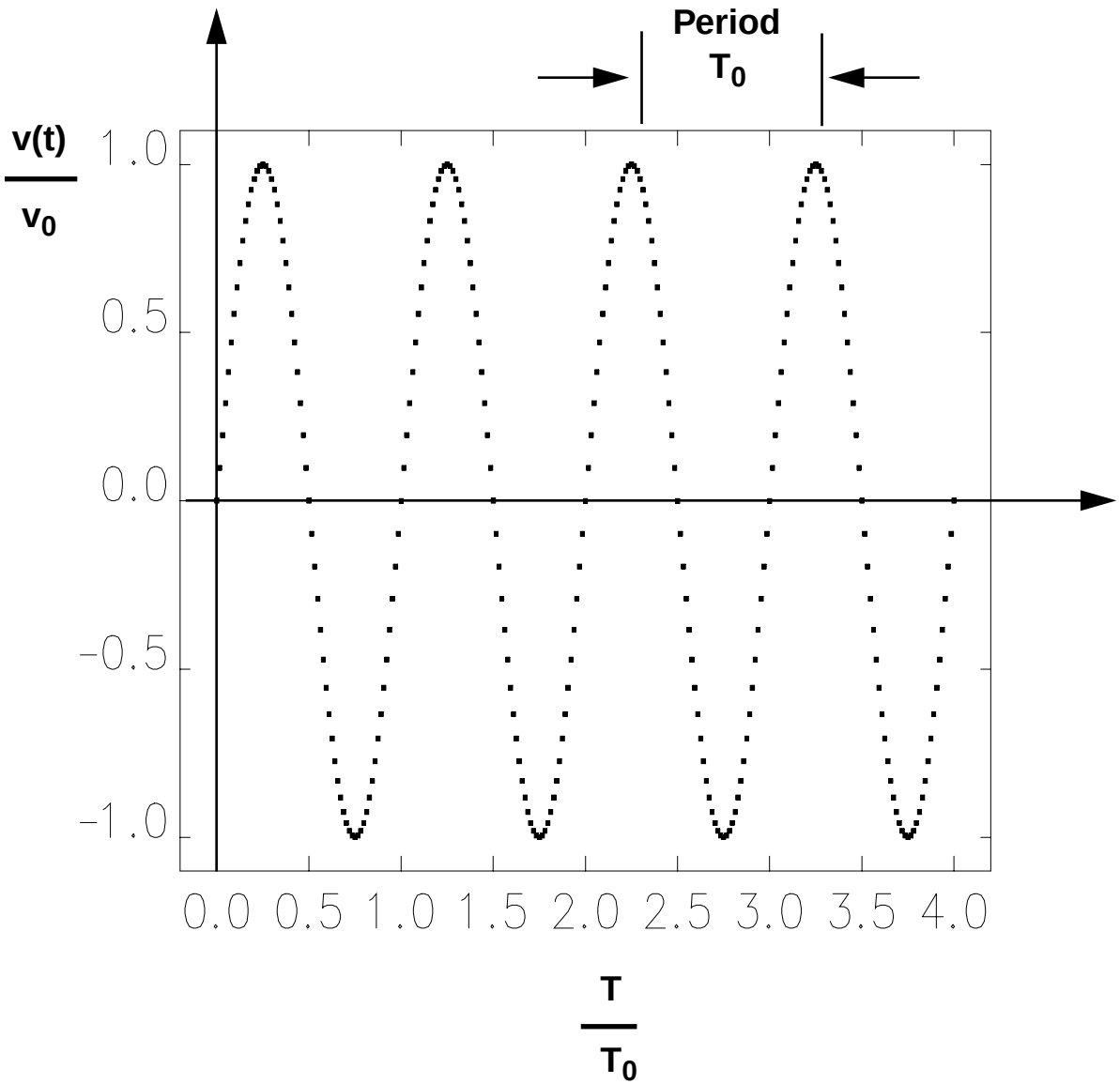
Frequency

Introduction to sampling theory and Fourier transforms
-- Digital Fourier transform
- Cross correlation function
- Auto correlation function
- Parseval's theorem

Accelerator characteristic frequencies
- RF frequency
- Bunch frequency
- Revolution frequency
- Sampling frequency
- Betatron frequencies
- Synchrotron frequency
- Line frequencies

Instrumentation
Spectrum Analyzer
Vector voltmeter
Vector Signal Analyzer
Network Analyzer

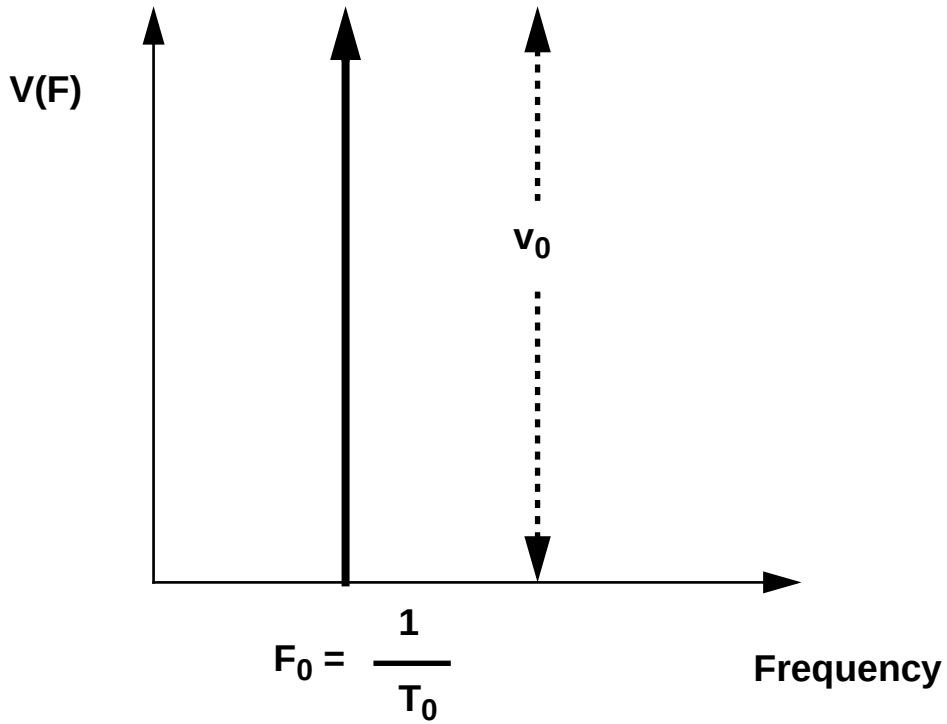
Simple Sine Wave



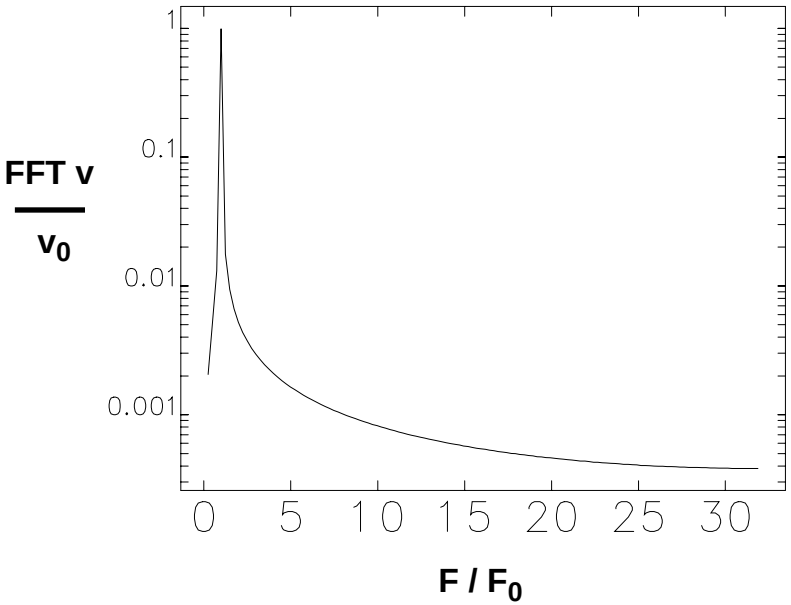
$$v(t) = v_0 \, \mathrm{Sin}\left\{ \frac{2\pi t}{T_0} \right\}$$

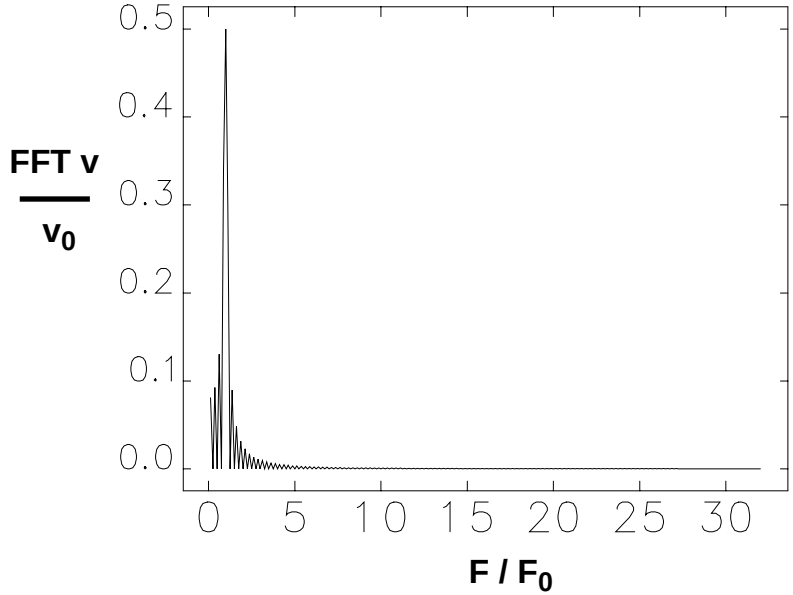
$$\text{Frequency } F_0 = \frac{1}{T_0}$$

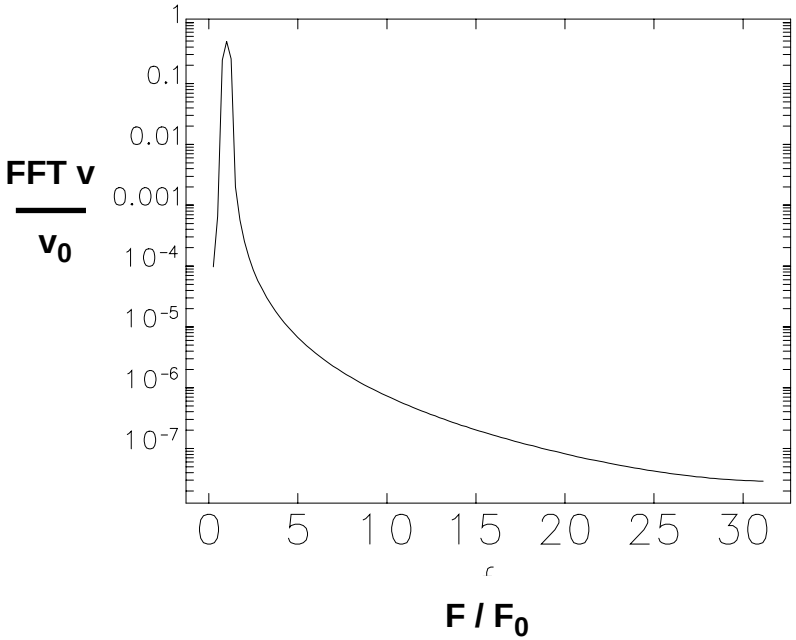
Simple Sine Wave Spectrum

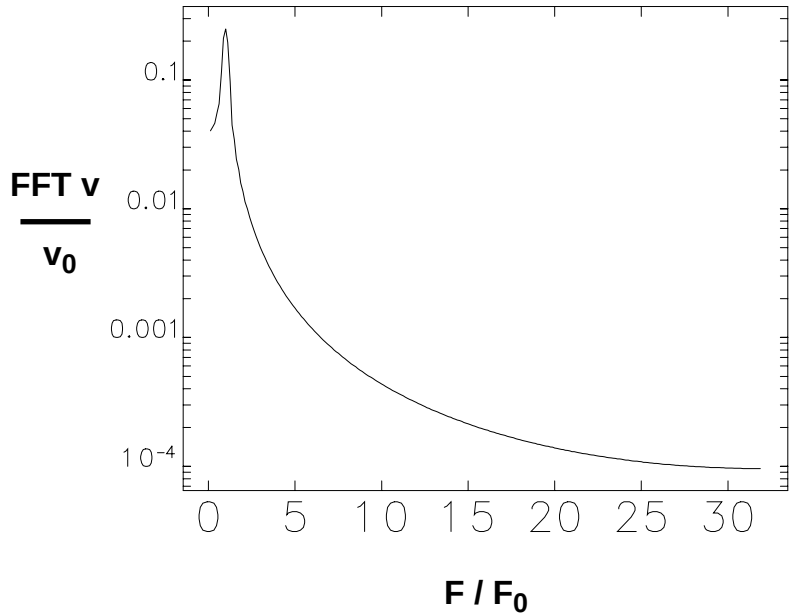


“Simple” Sine Wave Spectrum

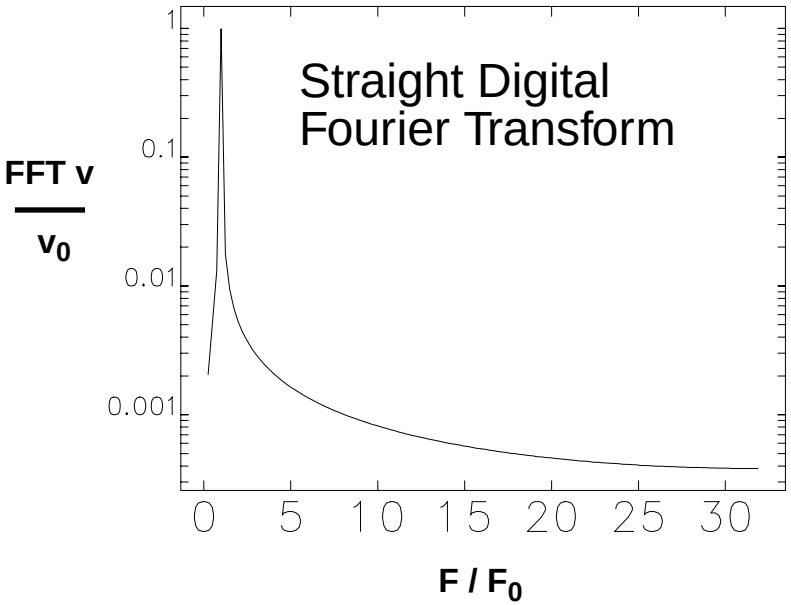


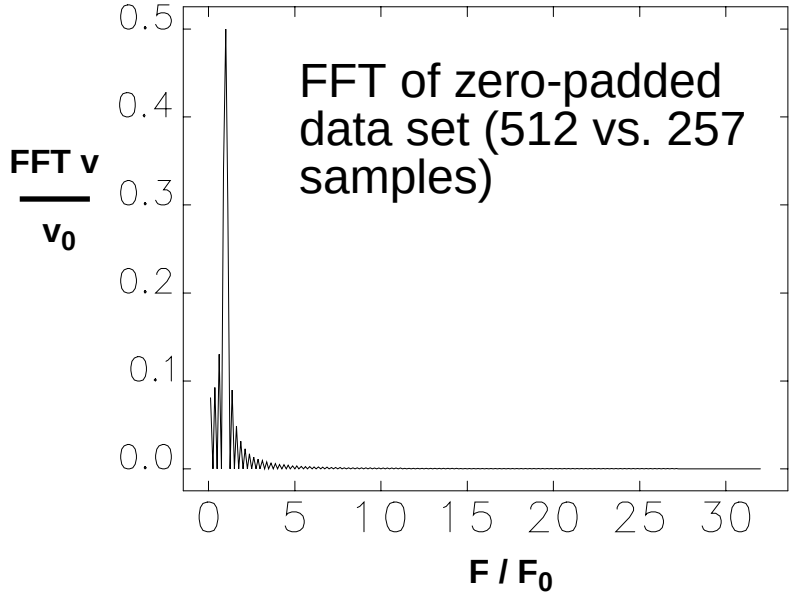


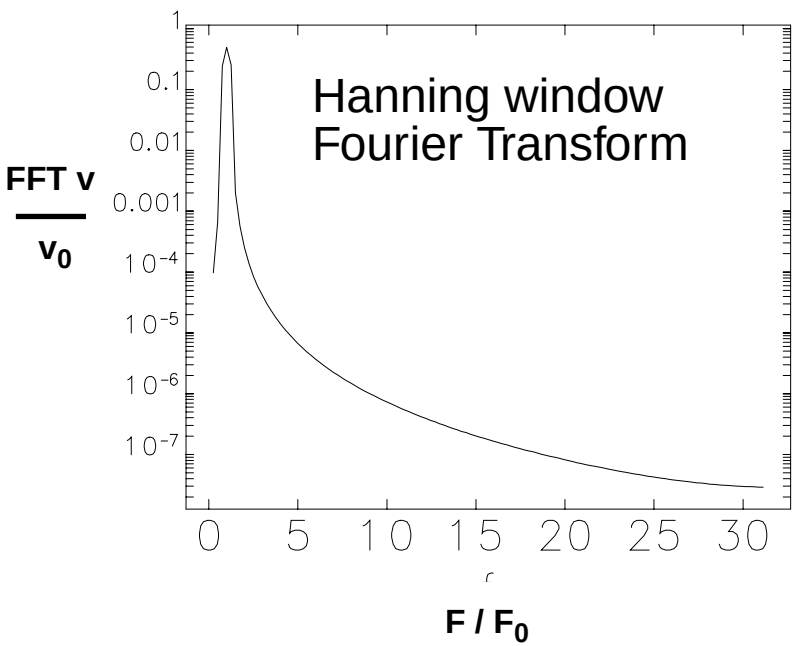


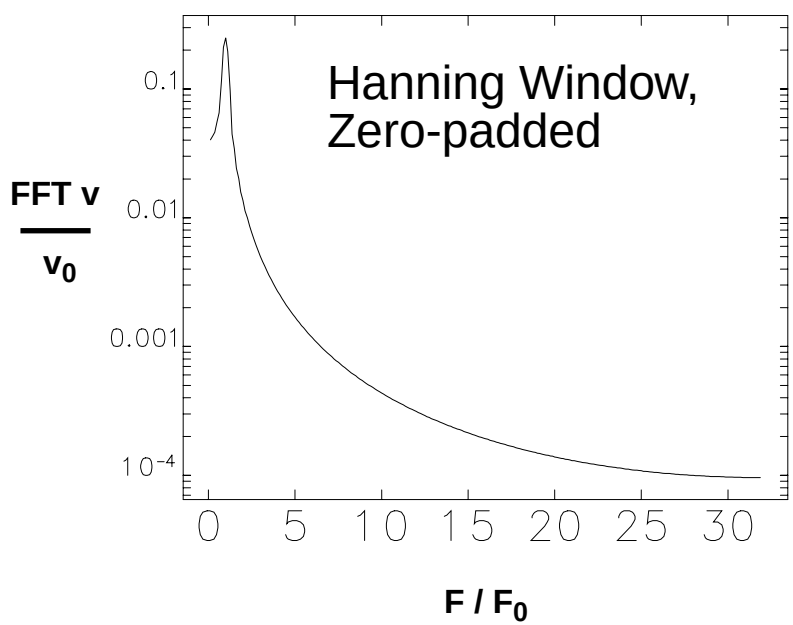


“Simple” Sine Wave Spectrum





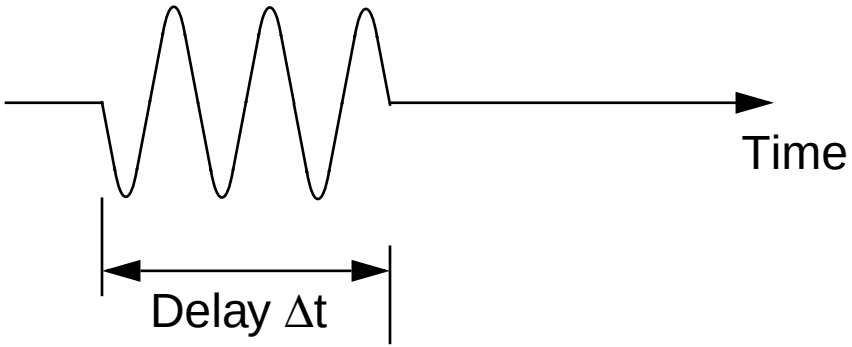




The Cross-Correlation Function

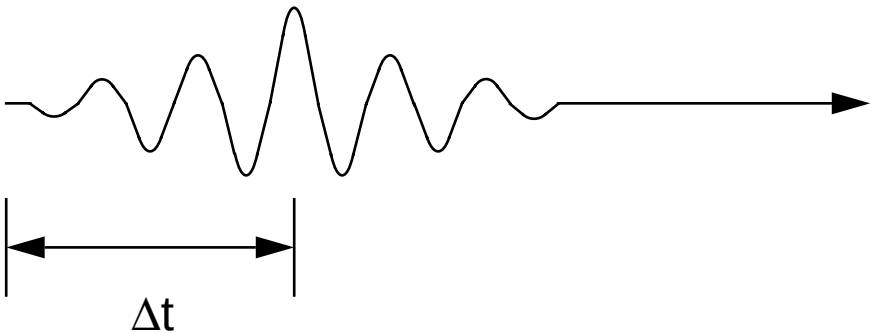
Used for radar range finding:

Transmitted Signal *x(t) :*



Received signal *y(t)* (∗1000) :

Correlation $r_{xy}(t)$:



$$r_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t+\tau)\,\mathrm{d}t$$

White Noise

Noise Signal One *x(t) :*

Uncorrelated Noise Signal Two *y(t)* :

Cross-correlation $r_{xy}(t) = 0$, (by definition of white noise)

Autocorrelation of White Noise

Noise Signal One *x(t) :*

Copy of Noise Signal One *x(t) :*

Autocorrelation function $r_{xx}(t) = \int_{-\infty}^{\infty} x(\tau)x(\tau+t)_d\tau$ = Const. $*$ δ(t)

Sampled Data

$$x_n = x(t_n), \qquad t_n = n\Delta, \qquad n = 0, 1, 2, 3, \ldots \text{N-1}$$

$$y_n = y(t_n), \qquad t_n = n\Delta, \qquad n = 0, 1, 2, 3, \ldots \text{N-1}$$

Δ = Sampling period

Cross-correlation $r_{xy}(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_n y_{n+m}$

Autocorrelation $r_{xx}(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_n x_{n+m} = \langle x^2 \rangle \, \delta_{m0}$ For random noise

$r_{xx}(0) = \frac{1}{N}\sum_{n=0}^{N-1} x_n x_n = \langle x^2 \rangle$ in general <---

= mean square signal

(it's square root is the rms signal)

Discrete Fourier Transform Pair

$$X(f_k) = X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{2\pi i n k/\mathrm{N}} \qquad f_k = \frac{k}{N\Delta} = 2 f_c \frac{k}{N}, \qquad k = 0, 1, \ldots (N-1)$$

Nyquist Frequency $f_c = \dfrac{1}{2\Delta}$

$$x(t_n) = x_n = \sum_{k=0}^{N-1} X_k e^{-2\pi i n k/\mathrm{N}} \qquad t_n = n\Delta, \qquad n = 0, 1, 2, 3, \ldots (N-1)$$

Note: $X_0 = \dfrac{1}{N}\sum_{n=0}^{N-1} x_n$ $= \langle x \rangle$, the mean value of the signal x_n

Note: X_k and x_n have the same dimensions

DFT of Cross-correlation

$$R_{xy}(k) = \frac{1}{N}\sum_{n=0}^{N-1} r_{xy}(n)e^{2\pi ink/\mathrm{N}} = X_k^{*}\, Y_k$$

DFT of Autocorrelation

$$R_{xx}(k) = \frac{1}{N}\sum_{n=0}^{N-1} r_{xx}(n)e^{2\pi ink/\mathrm{N}} = X_k^{*}\, X_k = |\, X_k\,|^{\mathbf{2}}$$

Parseval's Theorem

$$< x^2 > = \boxed{\frac{1}{N}\sum_{n=0}^{N-1}\left|x_n\right|^2 = \sum_{k=0}^{N-1}\left|X_k\right|^2}$$ = Mean-square signal amplitude

$$= X_0^2 + \sum_{k=1}^{N-1}\left|X_k\right|^2 = < x >^2 + \sum_{k=1}^{N-1}\left|X_k\right|^2$$

$$\boxed{< x^2 > - < x >^2 = \sum_{k=1}^{N-1}\left|X_k\right|^2 = \text{The variance } \sigma_x^2 \ ,}$$

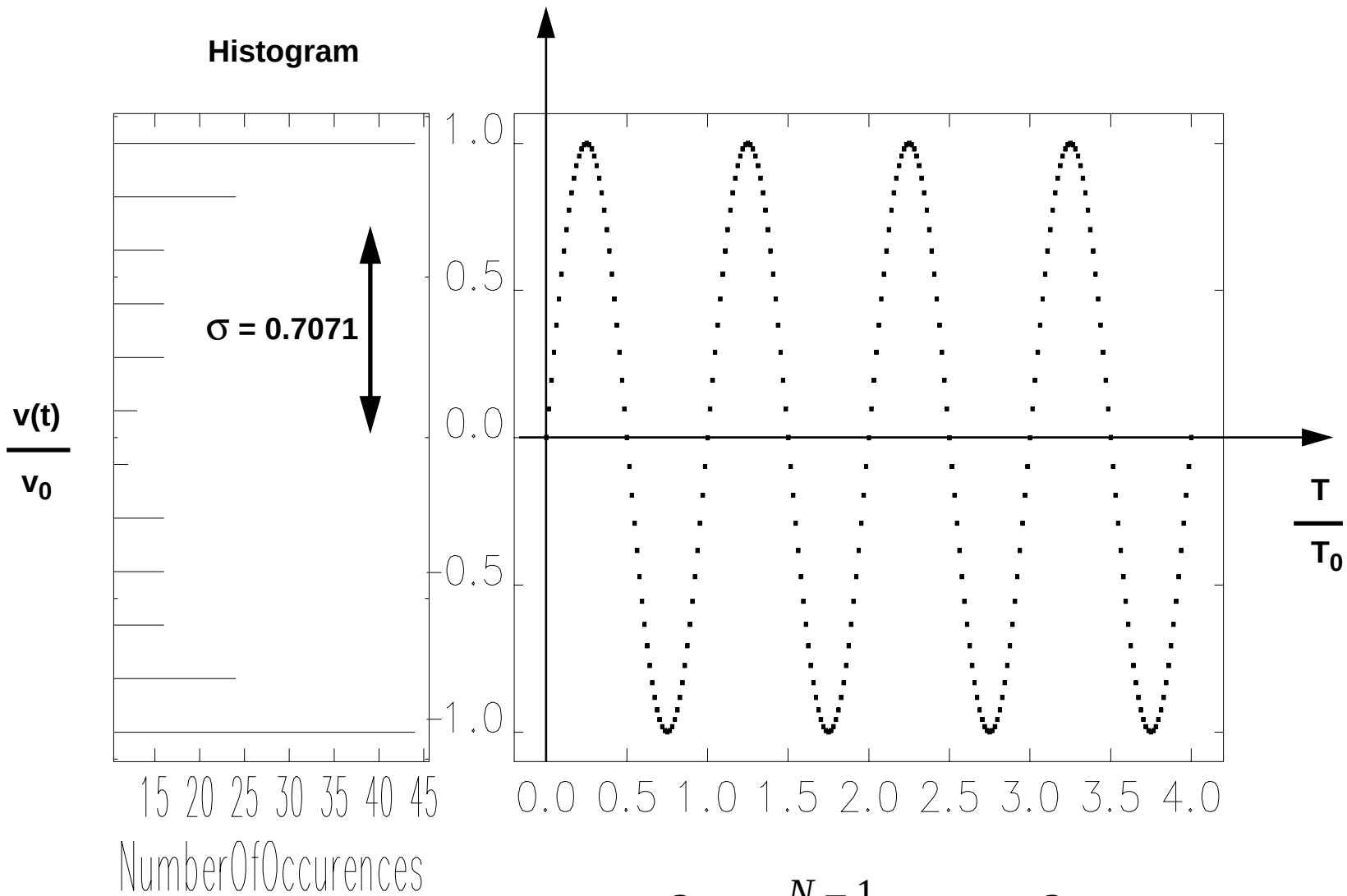
Can also be written*

$$\sigma_x^2 = \frac{1}{N}\sum_{n=0}^{N-1}(x_n - \langle x \rangle)^2$$

σ_x = the Standard Deviation

* The factor $\frac{1}{N}$ is commonly seen to be $\frac{1}{N-1}$, for obtuse reasons explained in, e.g., W. H. Press, etal, "Numerical Recipies in C," New York, Cambridge University Press, p. 611 - The definition given here makes more sense to me, and shouldn't make a hill of beans' difference

Statistics



$$\sigma_v^2 = \frac{1}{N}\sum_{n=0}^{N-1}(v_n - \langle v \rangle)^2$$

Noise Power

$$\sigma_x^2 = \frac{1}{N}\sum_{n=0}^{N-1}(x_n - \langle x\rangle)^2 = \langle x^2\rangle - \langle x\rangle^2 = \sum_{k=1}^{\mathrm{N-1}}\left|X_k\right|^2$$

Note:

$$X_{N\text{-}k} = X_{\text{-}k} = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-2\pi i n k/\mathrm{N}} = X_k^*$$

So

$$\boxed{\sigma_x^2 = \sum_{k=1}^{\mathrm{N-1}}\left|X_k\right|^2 = 2\sum_{k=1}^{\frac{N}{2}-1}\left|X_k\right|^2 + \left|X_{N/2}\right|^2}$$

Power Spectral Density

$$x_n = x(t_n), \qquad t_n = n\Delta, \qquad n = 0, 1, 2, 3, \ldots \text{ N-1}$$

$$X_k = X(f_k), \quad f_k = \frac{k}{N\Delta}, \qquad k = 0, 1, \ldots \frac{N}{2}$$

Variance of signal

$$\boxed{\sigma_x^2 = 2\sum_{k=1}^{\frac{N}{2}-1} \left|X_k\right|^2 + \left|X_{N/2}\right|^2}$$

The amount of mean square signal per unit of frequency

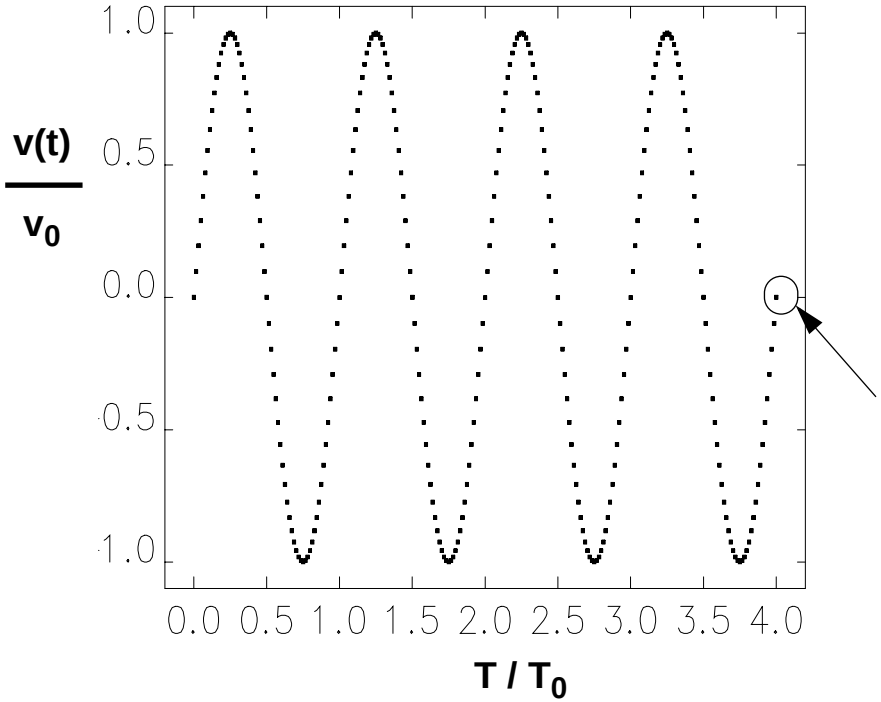
in the frequency span from $f_k - \frac{1}{2N\Delta}$ to $f_k + \frac{1}{2N\Delta}$ is approximately

$$P(f_k) = 2\left|X_k\right|^2/(\Delta f) = 2N\Delta\left|X_k\right|^2, \qquad k = 1, 2, \ldots \frac{N}{2} - 1$$

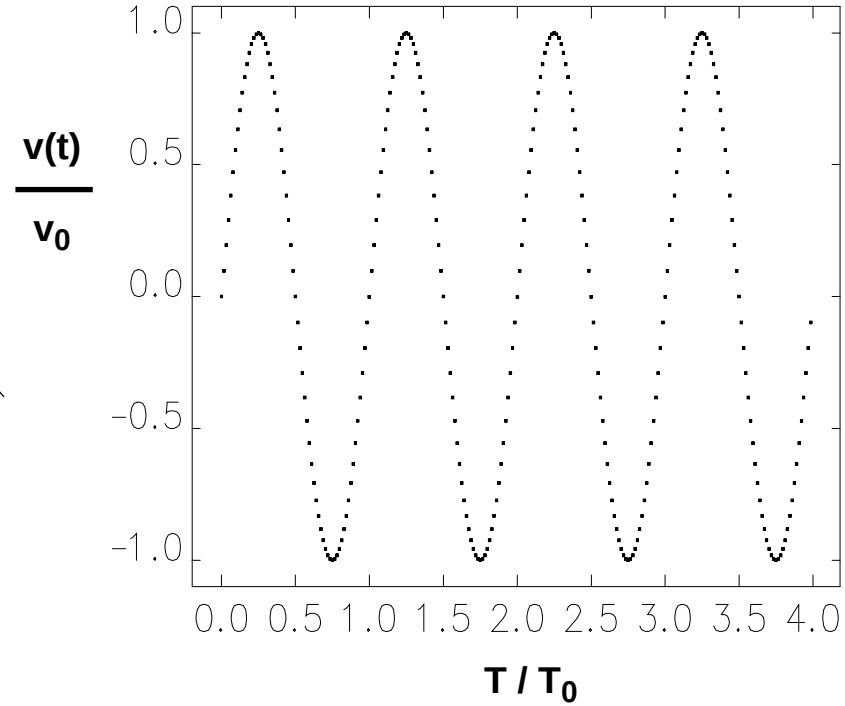
$$= N\Delta\left|X_{N/2}\right|^2, \qquad k = \frac{N}{2}$$

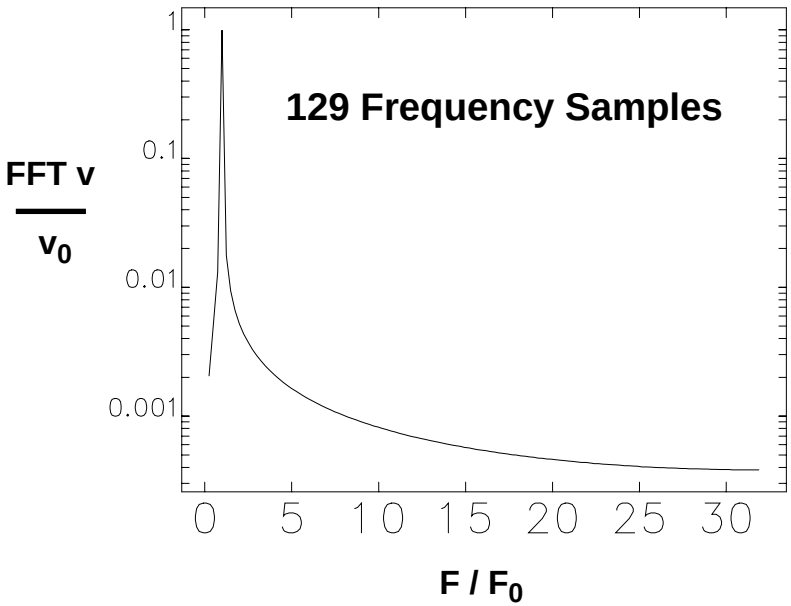
Note: The power spectral density P is measured in units = (dimension of x)^2 / frequency

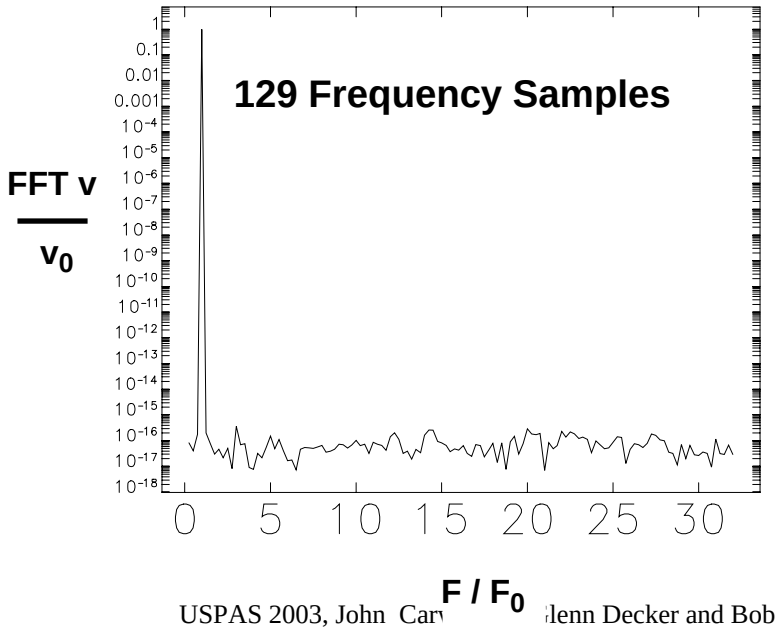
257 Data Points,
σ = **0**.7071067811865476

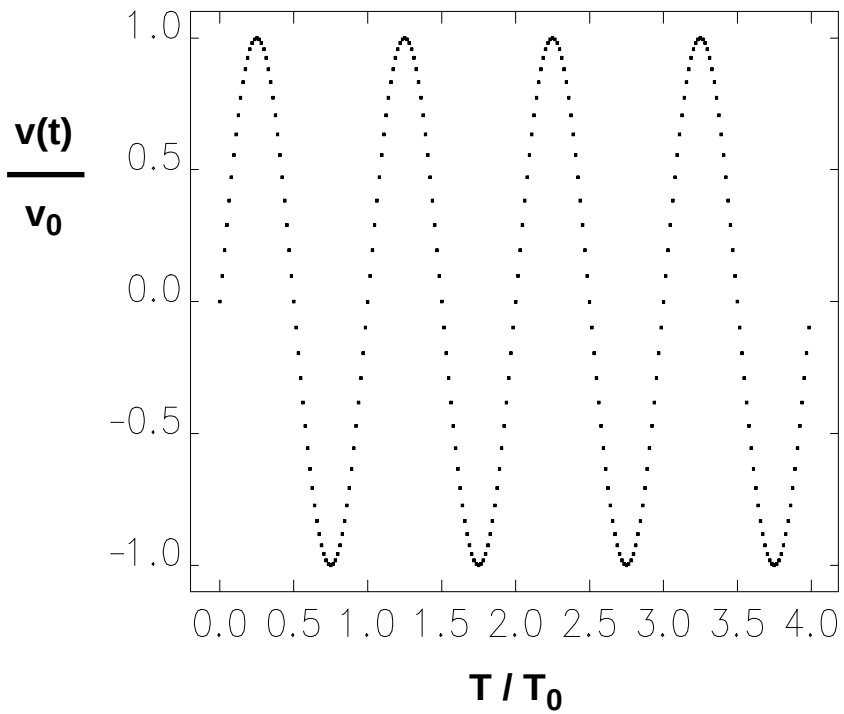


256 Data Points,
σ =7.084919084320763









$$\frac{\mathbf{v}(t_n)}{\mathbf{v_0}} = \frac{\mathbf{v}_n}{\mathbf{v_0}} = Sin\left(\frac{2\pi t_n}{T_0}\right) \qquad t_n = n\Delta$$

$$N = 256$$

Exactly 4 Periods = N = 256 Samples, so

$$T_0 = 64\Delta \longrightarrow \frac{2\pi t_n}{T_0} = 2\pi\frac{n}{64}$$

$$\frac{\mathbf{v}_n}{\mathbf{v_0}} \equiv v_n = \frac{e^{i\left(\frac{2\pi n}{64}\right)} - e^{-i\left(\frac{2\pi n}{64}\right)}}{2i}$$

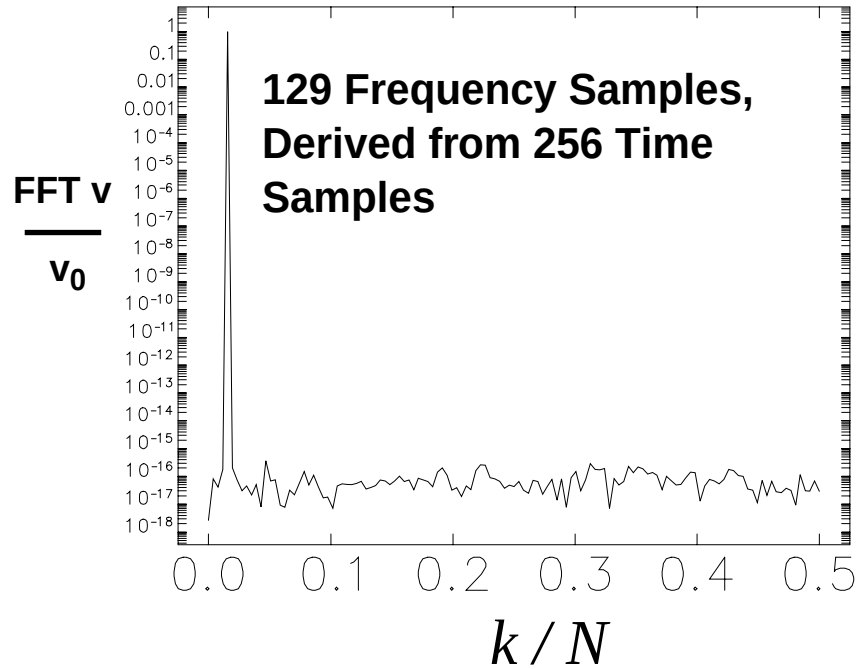
$$V_k = \frac{1}{N}\sum_{n=0}^{N-1} v_n e^{2\pi i n k/\mathrm{N}} = \frac{1}{N}\sum_{n=0}^{N-1}\left[\frac{e^{i\left(\frac{2\pi n}{256}\right)4} - e^{-i\left(\frac{2\pi n}{256}\right)4}}{2i}\right] e^{i\left(\frac{2\pi n}{256}\right)k}$$

Orthogonality Condition:

$$\sum_{n=0}^{N-1} e^{-i\left(\frac{2\pi n}{N}\right)k} e^{i\left(\frac{2\pi n}{N}\right)m} = N\delta_{k,m}$$

$$0 \le k \le N-1 \qquad \delta_{k,m} = \begin{cases} 1, (k = m) \\ 0, \text{Otherwise} \end{cases}$$

$$0 \le m \le N-1$$



so

$$V_k = \left(-\frac{\delta_{k,4}}{2i} + \frac{\delta_{-k,4}}{2i}\right) = \left(-\frac{\delta_{k,4}}{2i} + \frac{\delta_{N-k,4}}{2i}\right)$$

$$f_k = \frac{k}{N\Delta} \quad , \quad \Delta = \text{sampling period}$$

“Simple” Sine Wave Spectrum

$$V_k = -\frac{\delta_{k,4}}{2i}$$

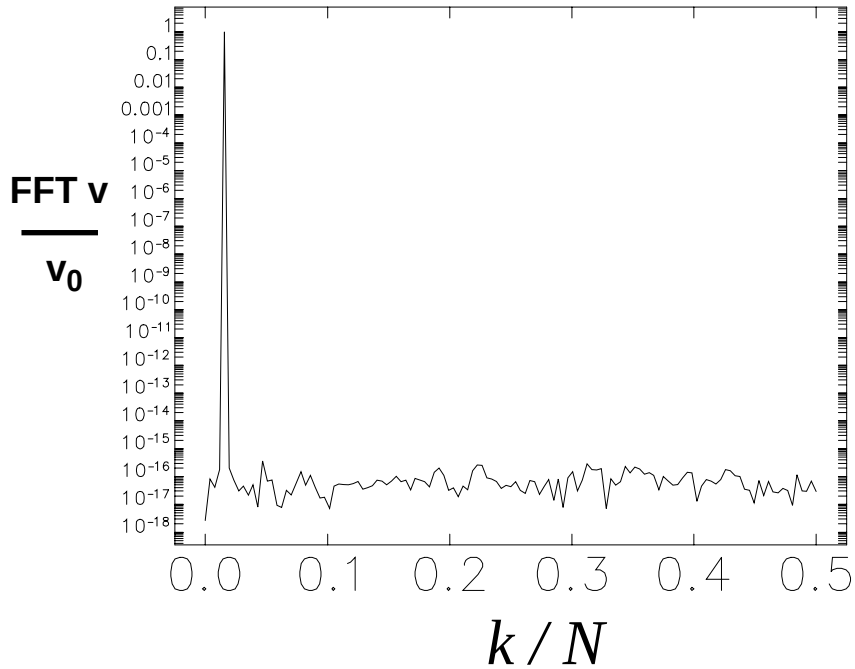
$$0 \le k \le \frac{N}{2}$$

$$\sigma_v^2 = 2\sum_{k=1}^{\frac{N}{2}-1} \left|V_k\right|^2 + \left|V_{N/2}\right|^2 = \frac{1}{2}$$

$$\longrightarrow \quad \sigma_v = \frac{1}{\sqrt{2}} = 0.7071067811865477$$

$$P(f_k) = 2N\Delta \left|V_k\right|^2 = \frac{N\Delta}{2}\delta_{k,4}$$

$$\sum_{k=1}^{\frac{N}{2}-1} P(f_k)\Delta f = \sum_{k=1}^{\frac{N}{2}-1} P(f_k)\frac{1}{N\Delta} = \frac{1}{2}$$



$$f_k = \frac{k}{N\Delta}$$

$$V_k = -\frac{\delta_{k,4}}{2i} \qquad 0 \le k \le \frac{N}{2}$$

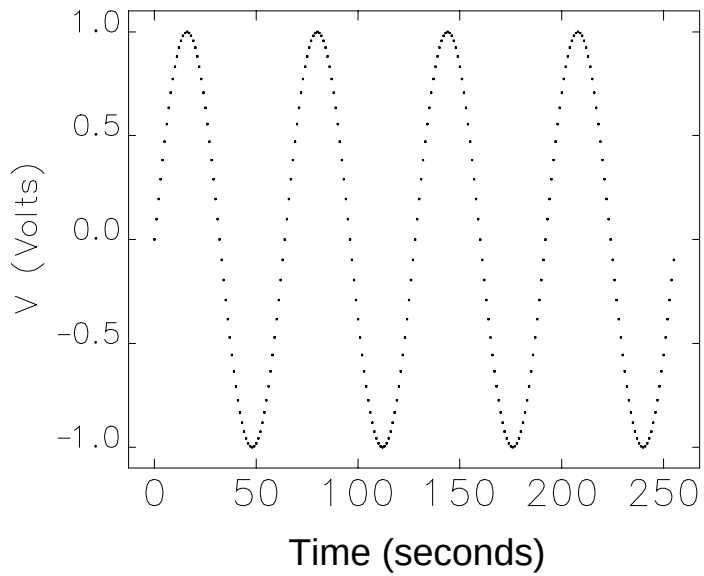
$$\sigma_v^2 = 2\sum_{k=1}^{\frac{N}{2}-1} |V_k|^2 + |V_{N/2}|^2 = \frac{1}{2}$$

$$\sqrt{2}|V_k| = \frac{1}{\sqrt{2}}\delta_{k,4} = \text{RMS Signal}$$

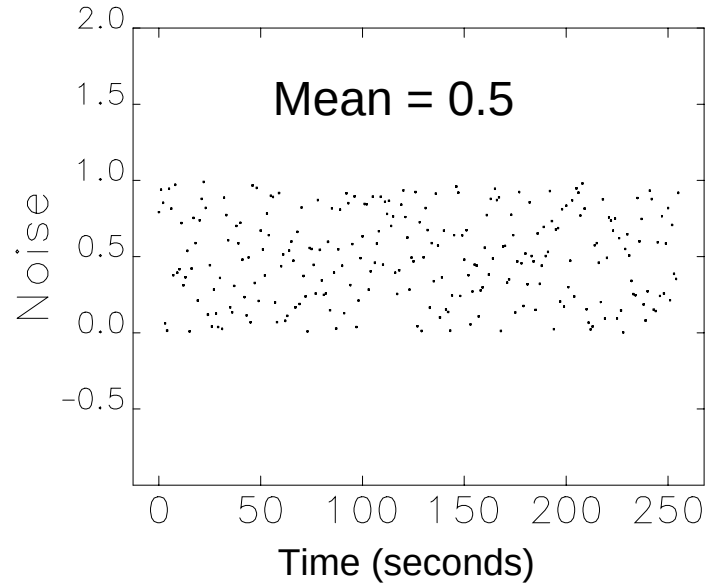
Finally $\dfrac{\text{FFT v}}{\text{v}_0} = \sqrt{2} * \text{RMS Signal} = 2|V_k| = \delta_{k,4} = \text{Peak Signal}$

A (slightly) more realistic example

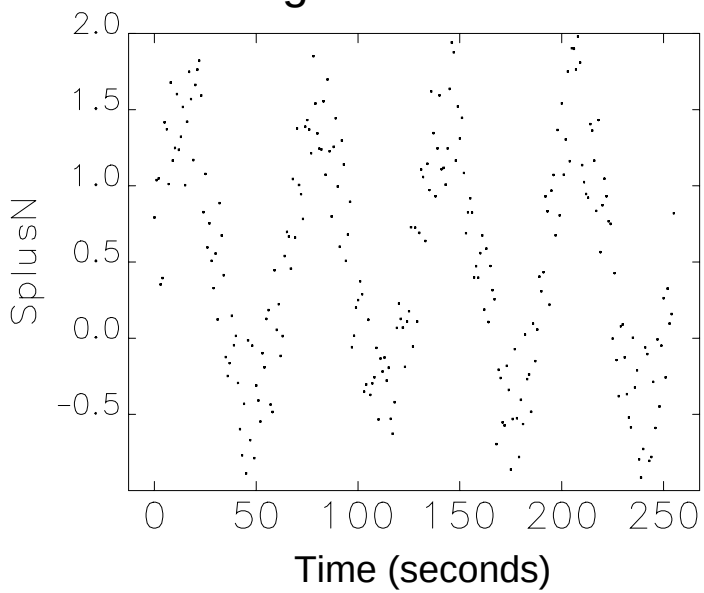
Signal



Noise



Signal + Noise

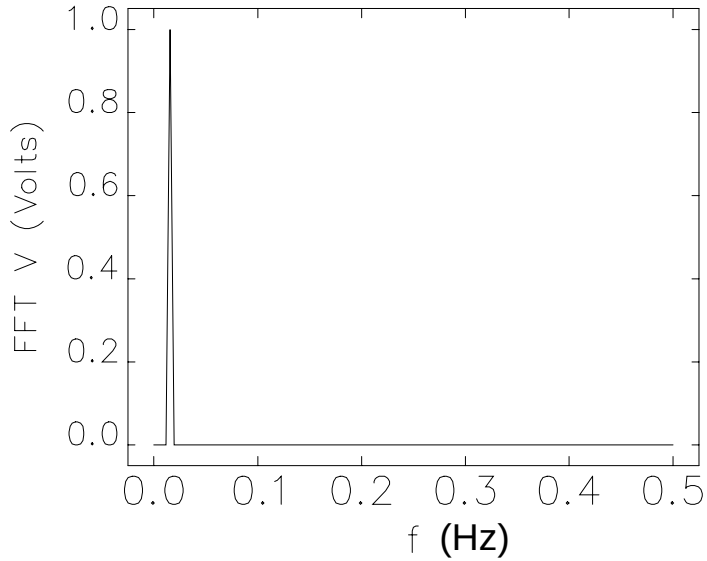


VMean (Volts) = -6.245005e-17
VRms (Volts) = 7.071068e-01
VStDev (Volts) = 7.084919e-01

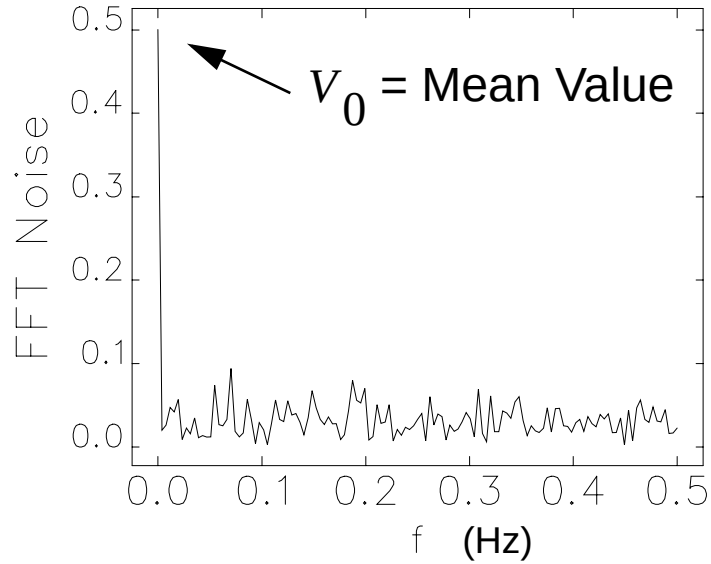
NoiseMean = 5.005493e-01
NoiseRms = 5.769418e-01
NoiseStDev = 2.874632e-01

SplusNMean = 5.005493e-01
SplusNRms = 8.932156e-01
SplusNStDev = 7.412359e-01

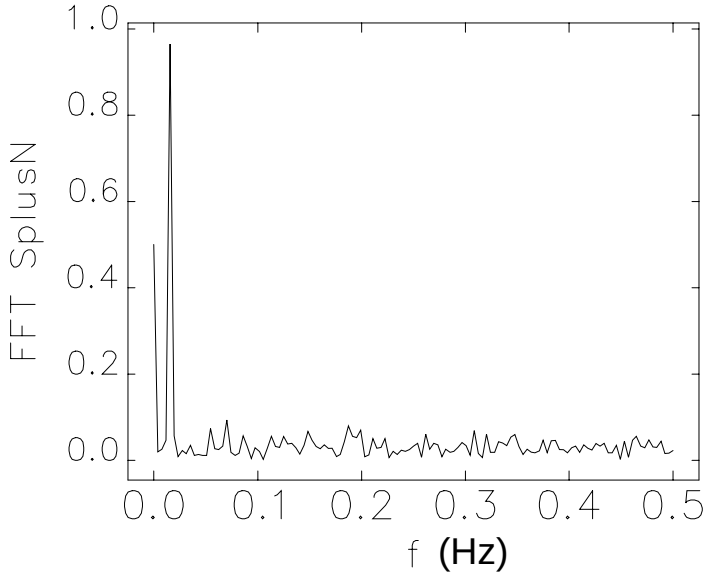
Peak Signal Spectrum $2|V_k|$



Peak Noise Spectrum



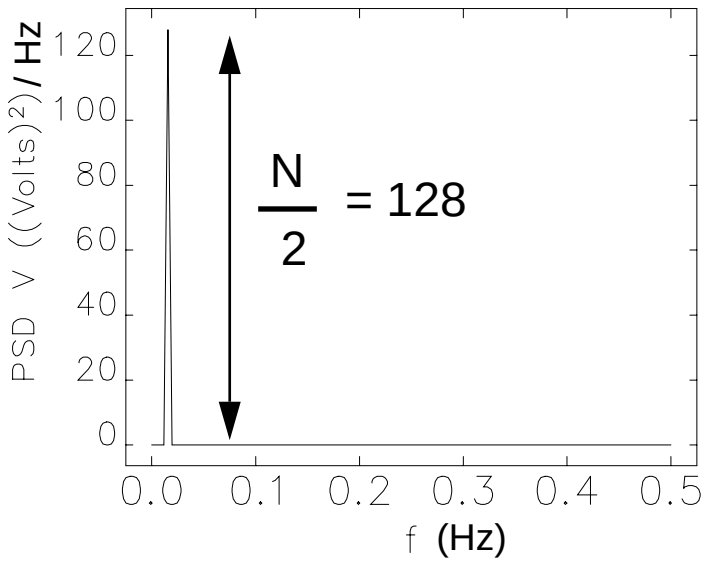
Signal + Noise



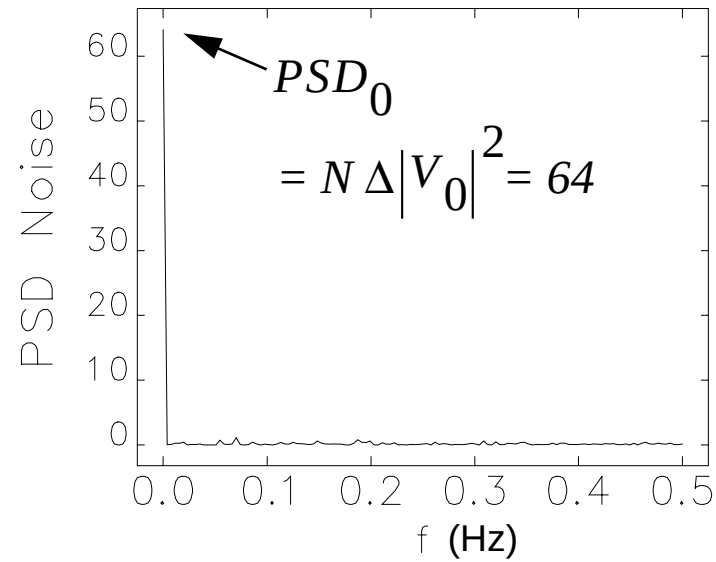
$$f_k = \frac{k}{N\Delta}$$, Δ = sampling period
= 1 second for this example

Power Spectral Densities

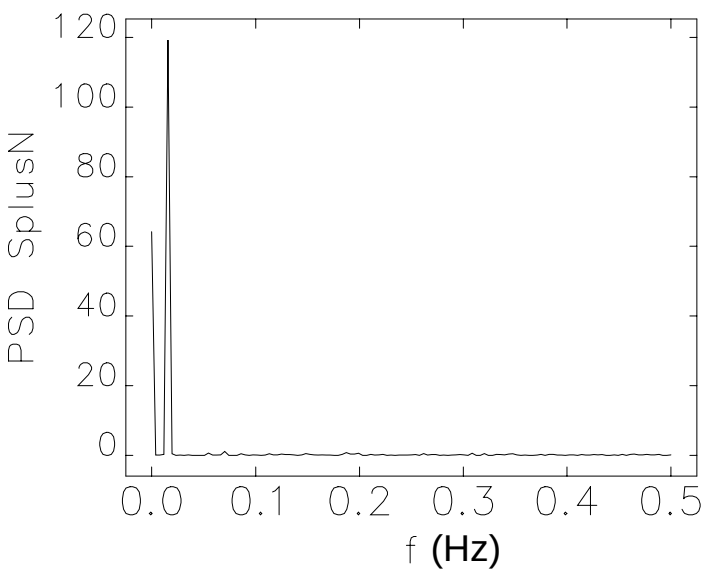
Signal



Noise



Signal + Noise

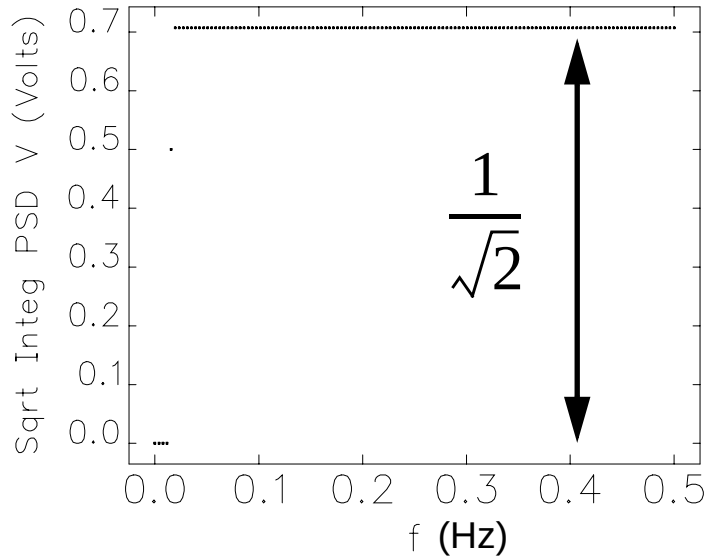


$$P(f_k) = 2N\Delta \left|V_k\right|^2, \quad \Delta = 1$$

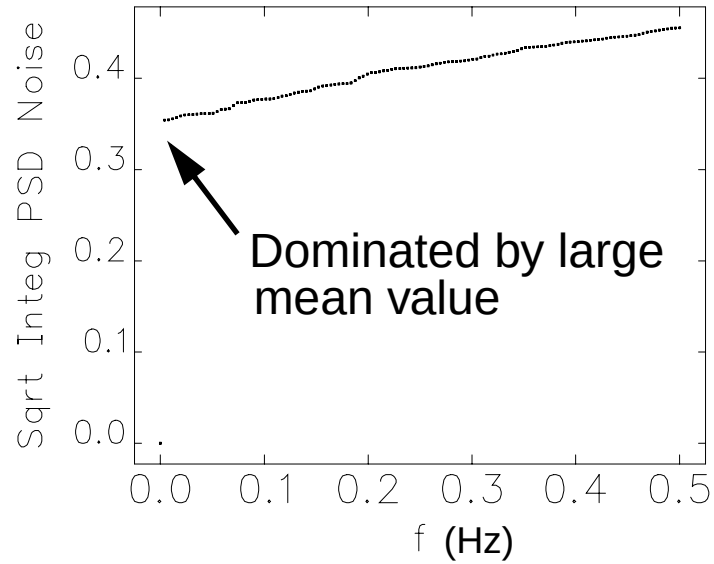
(except k = 0 and k = N / 2)

Integrated Power Spectral Densities

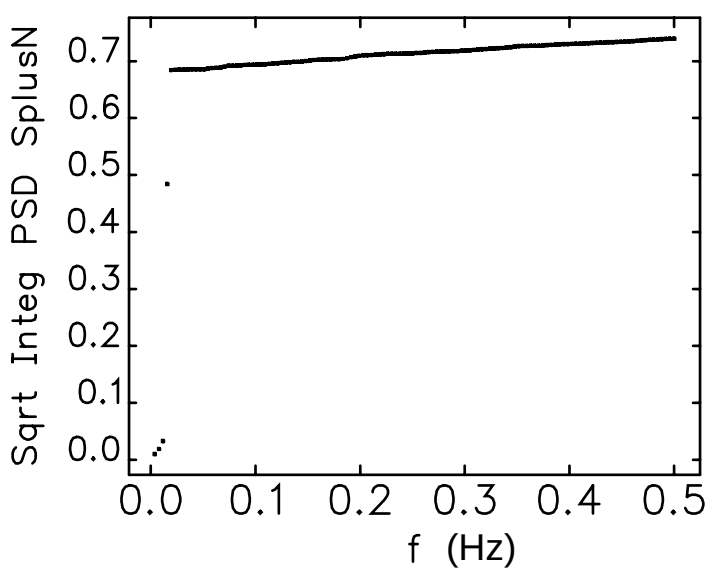
Signal



Noise



Signal + Noise



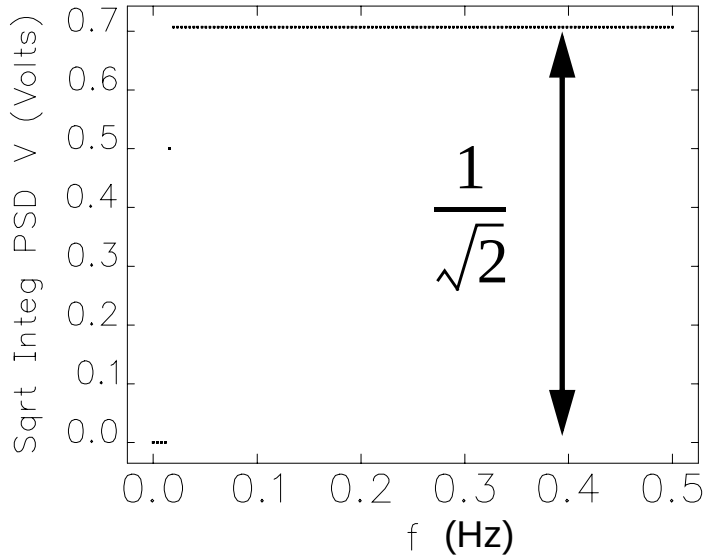
$$Vrms = \sqrt{\int_0^f P(f)\,df}$$

Note: $\int_0^f P(f)\,df$ *is shorthand for*

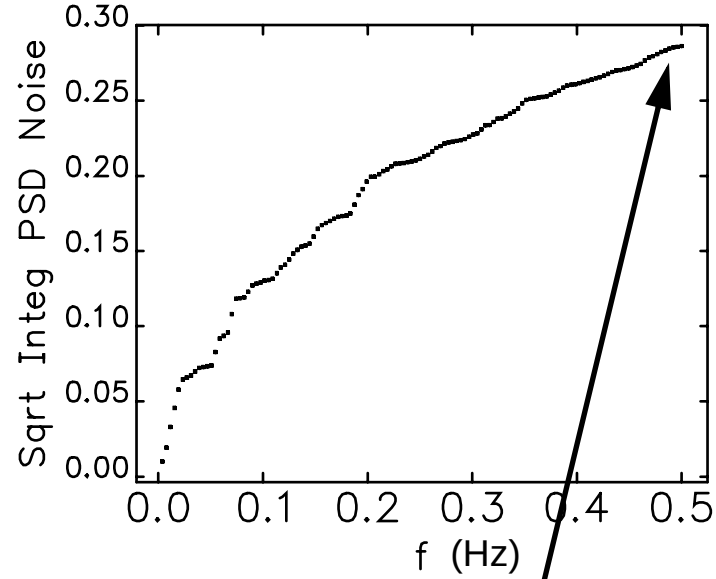
$$\sum_{k'=0}^{k} P(f_{k'})\Delta f$$

Integrated Mean-Suppressed Power Spectral Densities

Signal

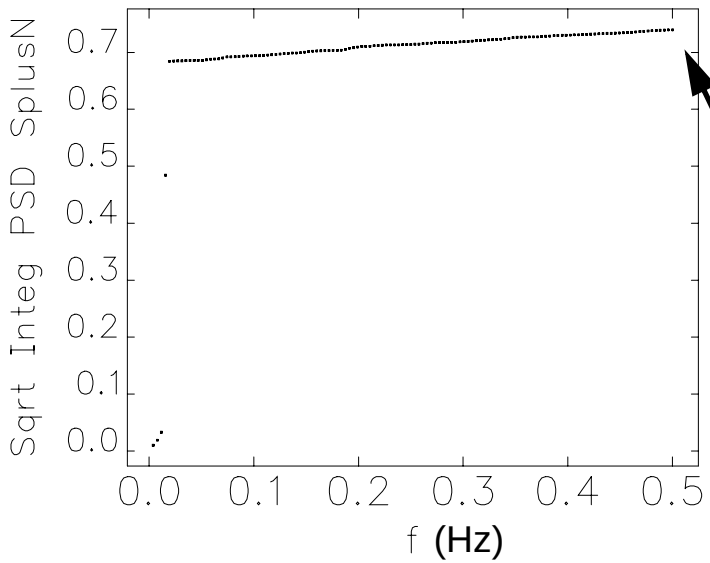


Noise



NoiseStDev = 2.874632e-01

Signal + Noise

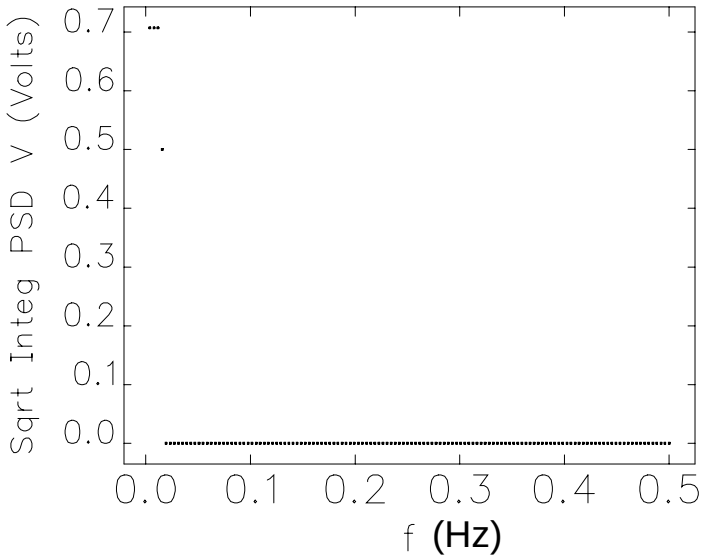


S+N StDev = 0.74 < (SignalStDev^2+NoiseStDev^2)$^{1/2}$ = 0.76

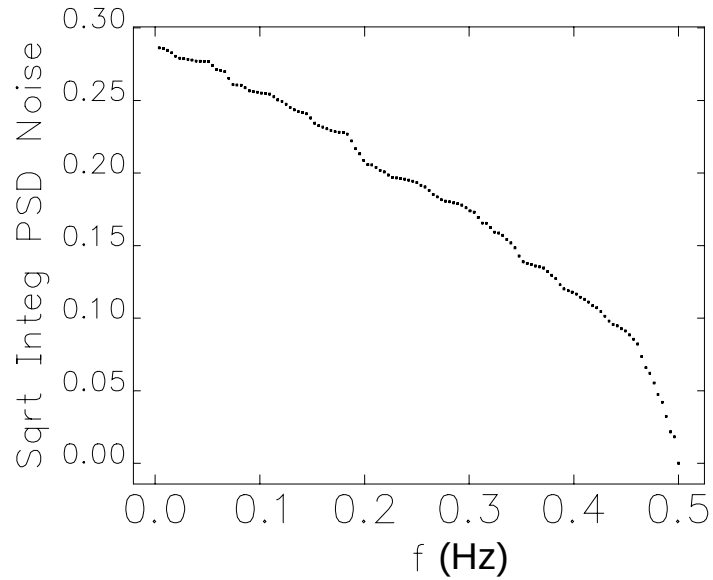
(Both Signal + Noise are non-Gaussian)

Reverse-Integrated Mean-Suppressed Power Spectral Densities

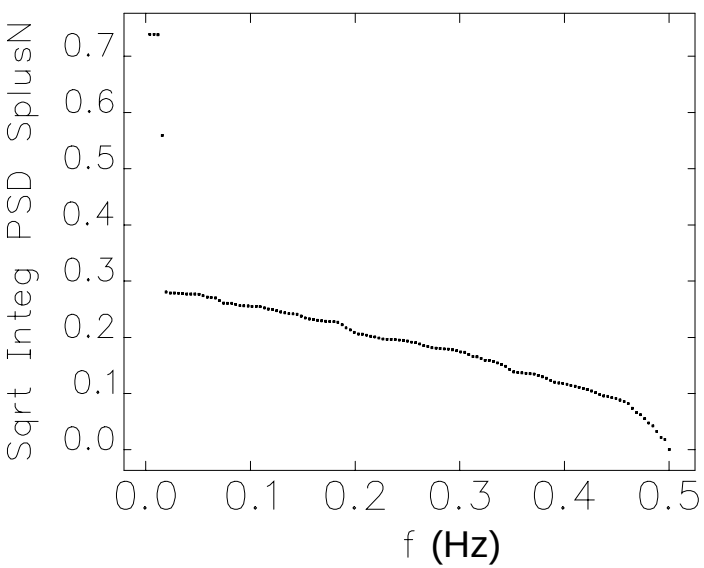
Signal



Noise



Signal + Noise



$$Vrms = \sqrt{\int_{f_c}^{f} P(f)\,df}$$

$$f < f_c = \frac{1}{2\Delta} = \frac{1}{2}\ \text{Hz}$$

Useful for resolving high frequency components in the presence of large signal at low frequency.

Homework Problem - Computer Lab

Given the power spectrum shown, together with the “voltage spectrum” (.xls file) -

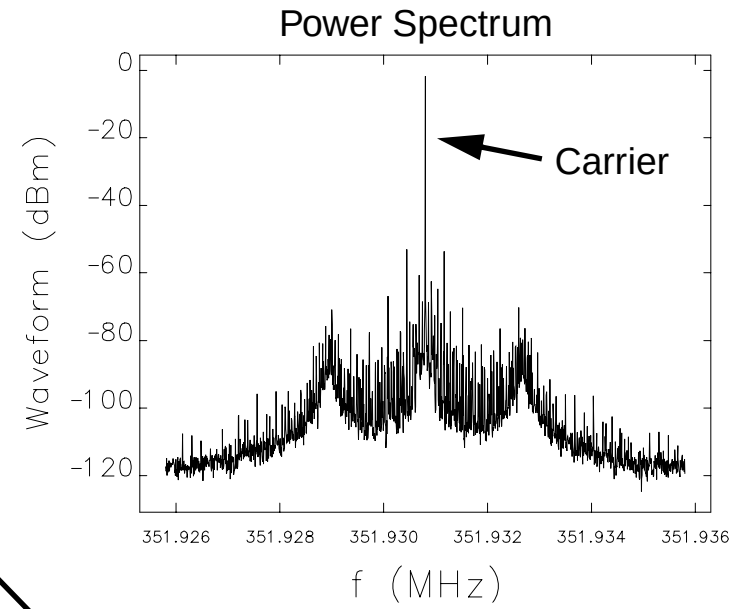
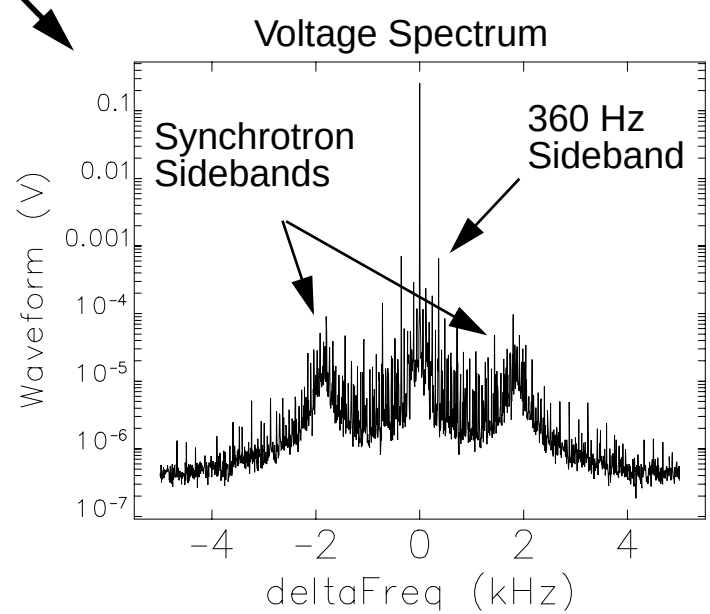
1) Is this a peak voltage spectrum, rms, or something else?
Hint 0 dBm = 1 mW rms into a 50 ohm load.

2) What is the relation between the given voltage spectrum and $|V_k|$?

3) Plot the power spectral density in (Volts^2 / Hz) vs. deltaFreq

4) What is the voltage variance and standard deviation in the full 10 kHz frequency span, not counting the carrier.?
Note - deltaFreq < 0 does not imply V_k -> V_k * = V_{-k}

5) What is the ratio of rms voltage in the upper 360 Hz sideband to the overall rms voltage corresponding to deltaFreq > 0 ?

6) What is the ratio of the rms voltage in the 360 Hz sidband to the rms voltage associated with the upper synchrotron sideband between 1.5 and 2.5 kHz.

7) Plot the square root of the reverse-integrated power spectral density from 5 kHz down to, but not including deltaFreq=0, in volts rms. Simpson rule integration is good enough.