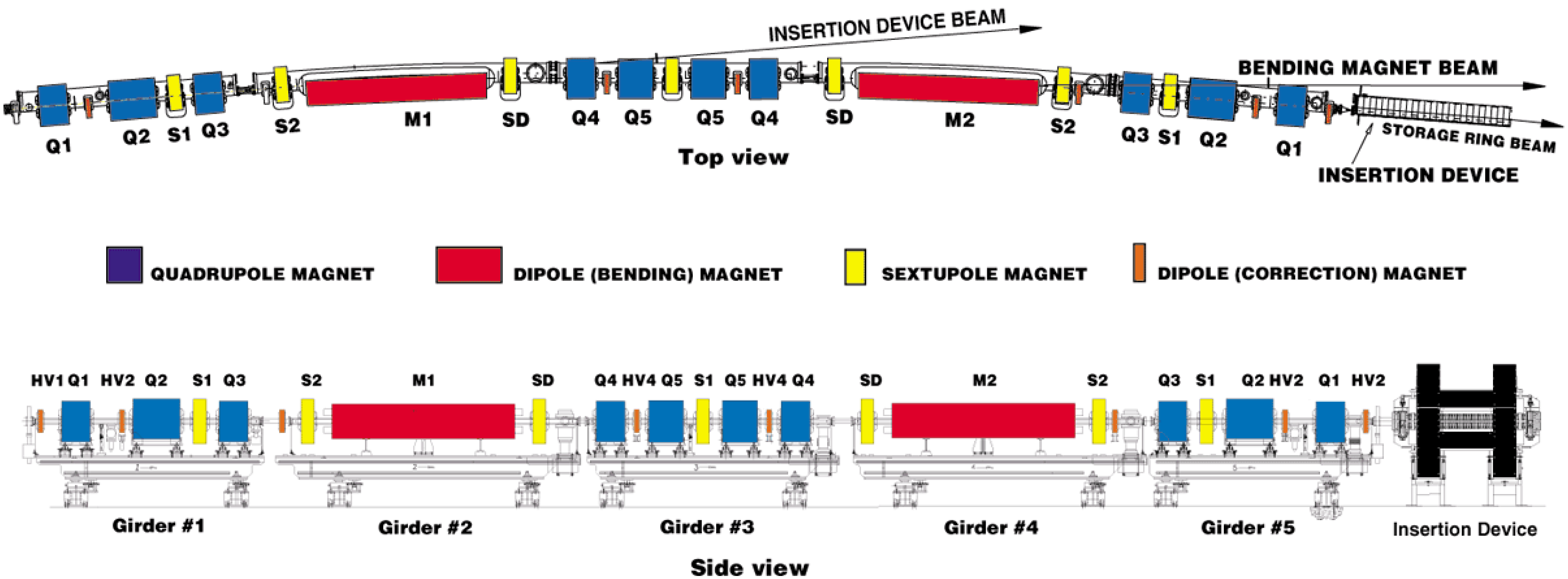
Essentials of Accelerator Physics



One of the 40 Sectors of the Advanced Photon Source

Essentials of Accelerator Physics

Single particle horizontal displacement relative to “the” equilibrium closed orbit:

$$x(s) = [W_x\ \beta_x(s)]^{1/2} \cos[\psi_x(s) - \psi_{0x}]$$

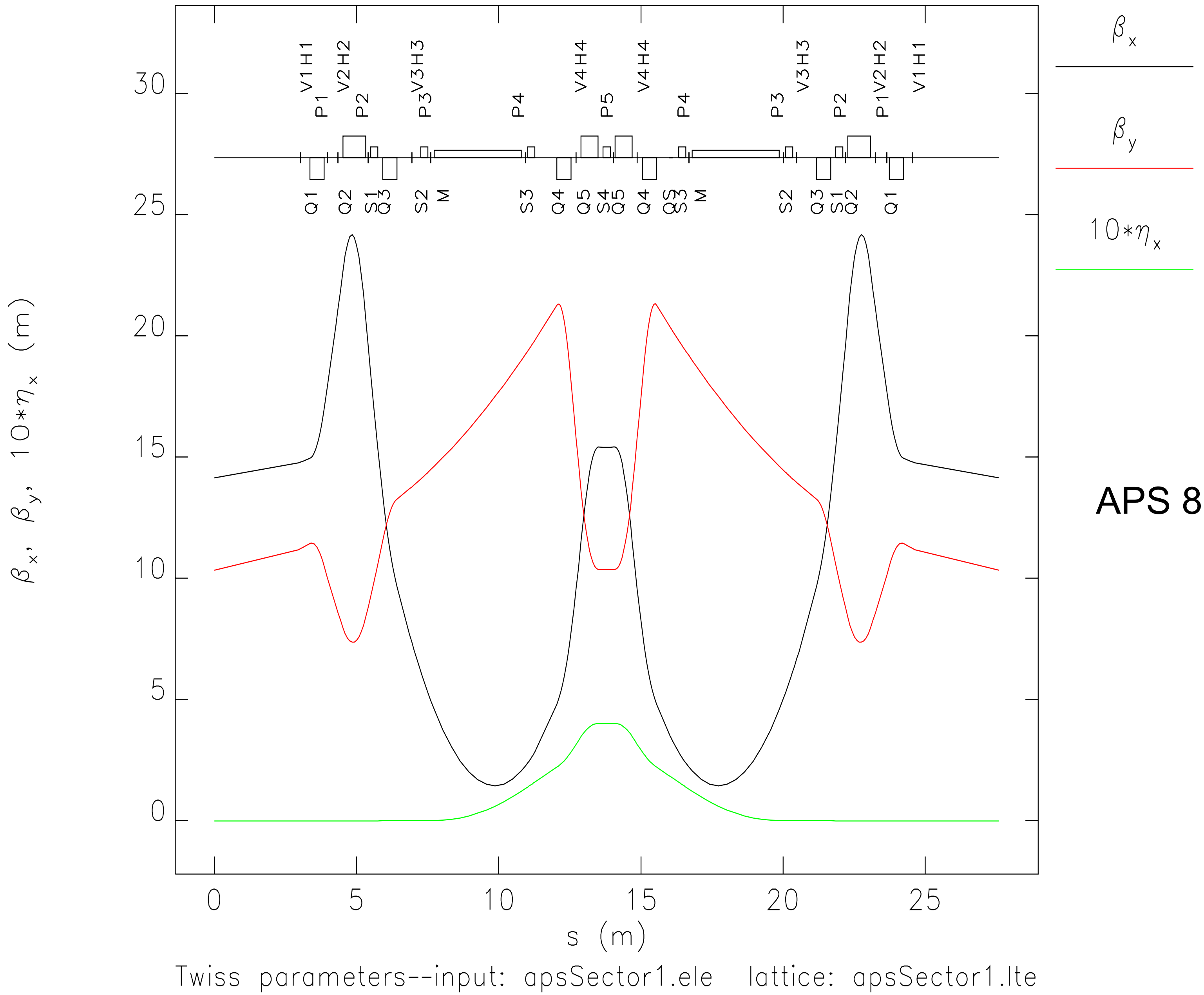
Single particle vertical displacement relative to “the” equilibrium closed orbit:

$$y(s) = [W_y\ \beta_y(s)]^{1/2} \cos[\psi_y(s) - \psi_{0y}]$$

Displacement of the equilibrium closed orbit for an off-energy particle:

$$\Delta x_E(s) = \eta(s)\,\frac{\Delta E}{E_0} = \eta(s)\,\delta$$

Double Bend Achromat Lattice - also known as Chasman-Green Lattice



APS 8.2 nm-rad

Essentials of Accelerator Physics

Twiss Parameters $\alpha(s)$, $\beta(s)$, $\gamma(s)$

$$\alpha(s) = -\beta'(s) / 2 \qquad\qquad \gamma(s) = \frac{1 + \alpha^2(s)}{\beta(s)}$$

All Twiss parameters are periodic functions:

$$\beta(s+L) = \beta(s)$$
$$\alpha(s+L) = \alpha(s)$$
$$\gamma(s+L) = \gamma(s)$$

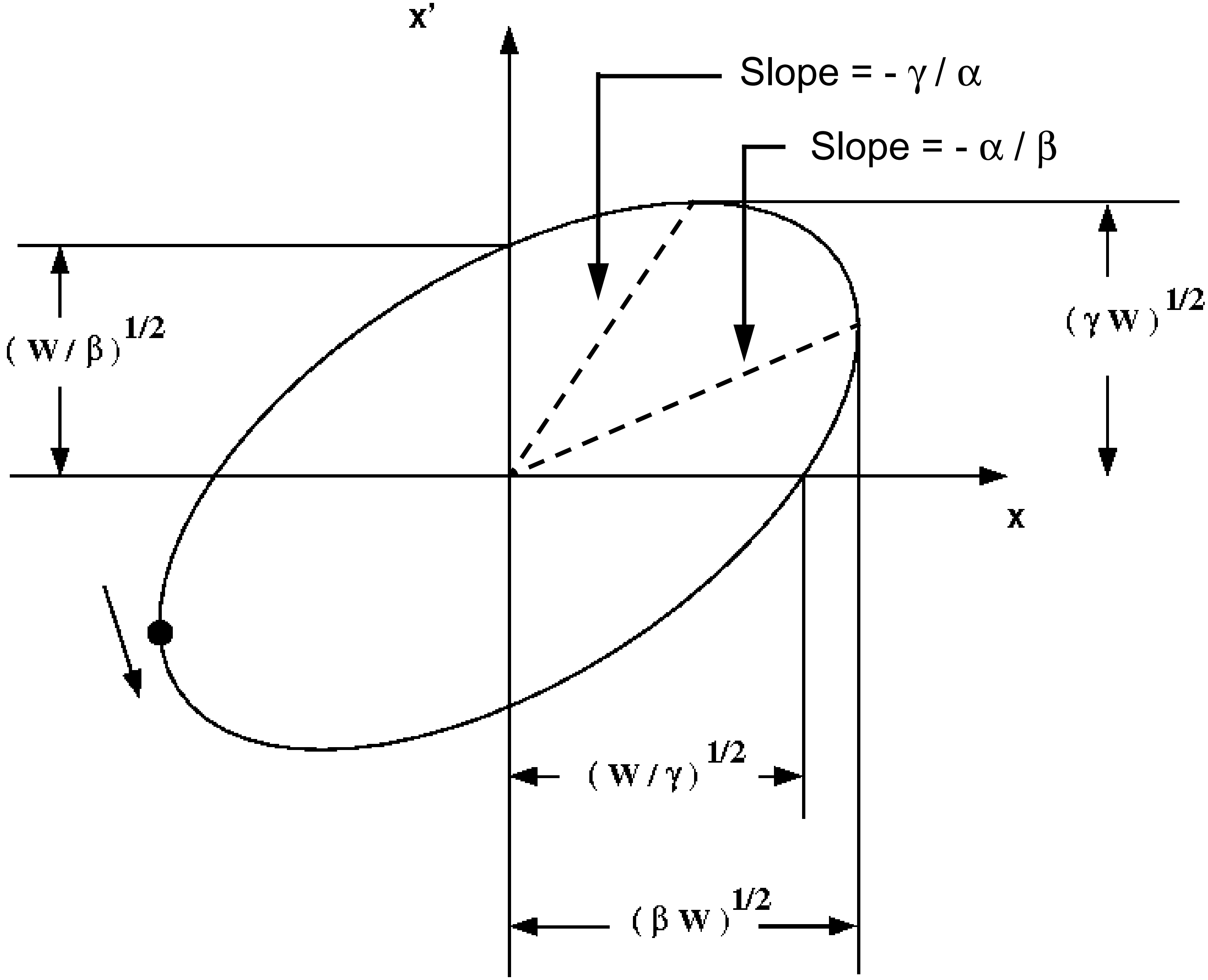
L = Ring "Circumference" = c T_{rev}

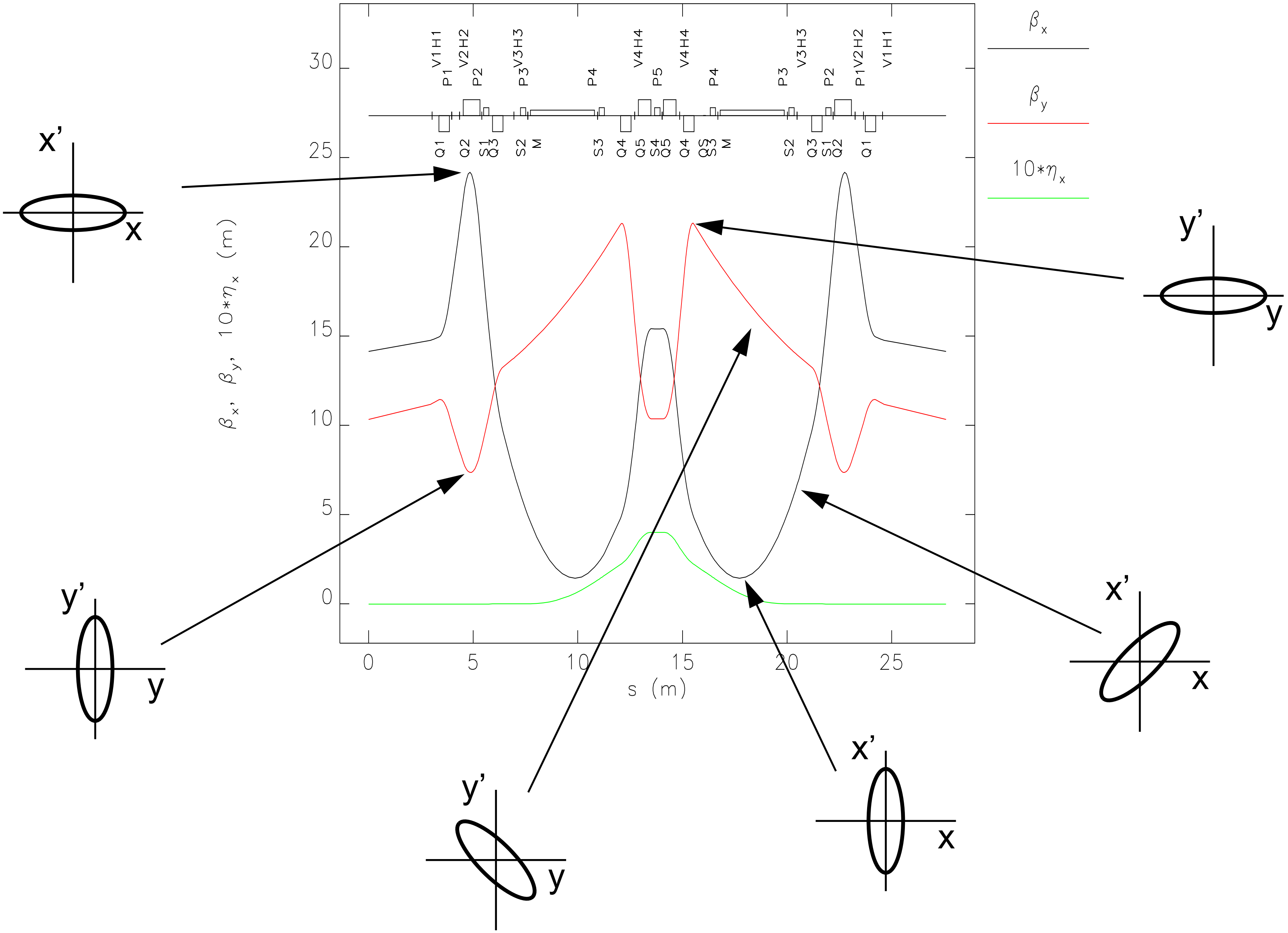
Courant-Snyder Invariant* $W_x = \gamma_x(s)\, x^2(s) + 2\, \alpha_x(s)\, x(s)\, x'(s) + \beta_x(s)\, x'(s)^2$

$$= [\, x^2 + (\, \alpha_x x + \beta_x x')^2\,] / \beta_x$$

* E.Courant, H.S. Snyder, Annals of Physics 3, 1-48 (1958)

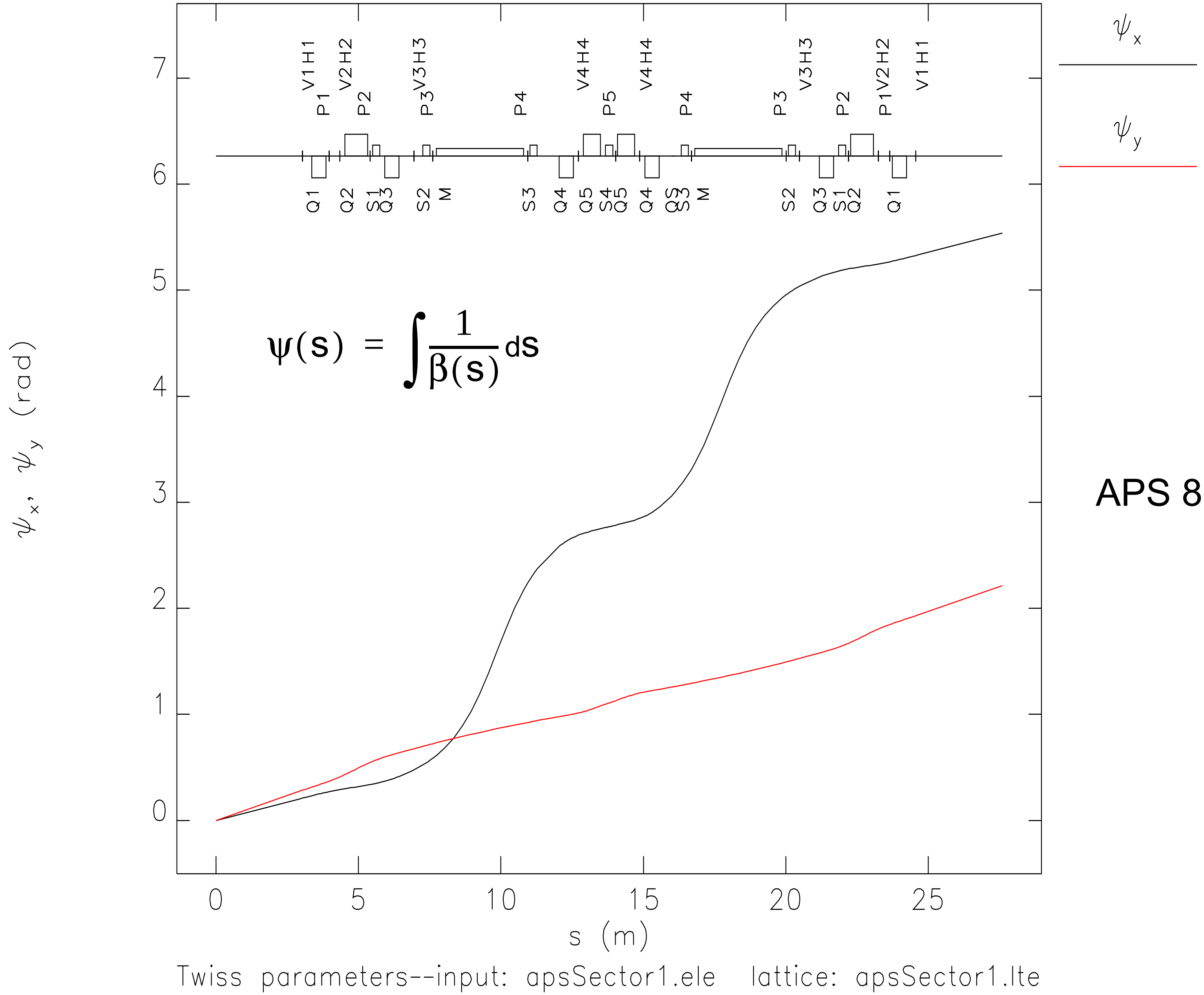
Particle Beam Phase Space





β_x
β_y
$10*\eta_x$
β_x, β_y, $10*\eta_x$ (m)
s (m)
x'
x
y'
y

Horizontal, Vertical Betatron Phases



APS 8.2 nm-rad

Betatron Tunes

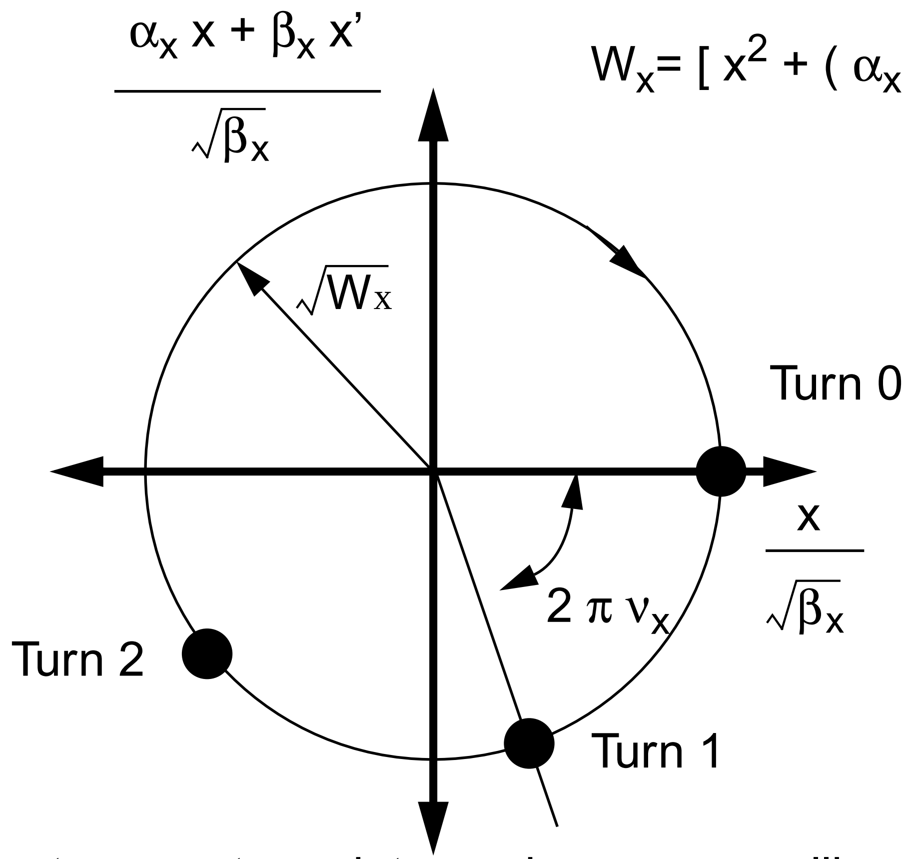
The betatron tune ν is the betatron phase advance in one circuit around the machine, divided by 2π, i.e. it is the number of betatron wavelengths once around the machine:

$$2\pi\nu_x = \psi_x(s+L) - \psi_x(s) = \int_s^{s+L} \frac{1}{\beta_x(s)}\,ds$$

$$2\pi\nu_y = \psi_y(s+L) - \psi_y(s) = \int_s^{s+L} \frac{1}{\beta_y(s)}\,ds$$

Normalized Electron Phase Space at Fixed Location s

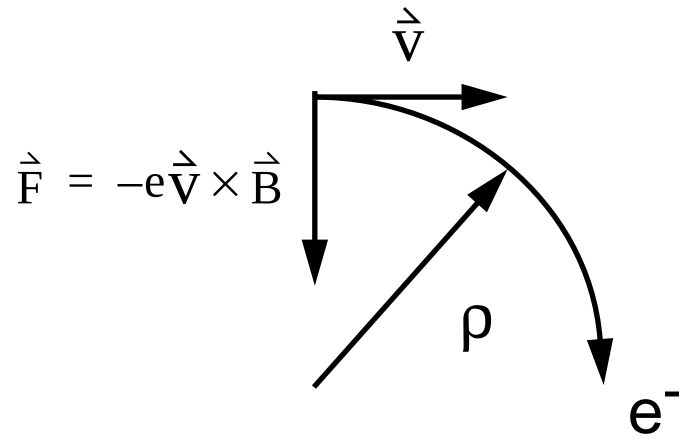
$$W_x = [\, x^2 + (\, \alpha_x x + \beta_x x')^2 \,] / \beta_x$$



$\dfrac{d\,\beta_x(s)}{d\,s} = -\,2\,\alpha_x = 0$ at symmetry points -> phase space ellipse is upright, so

$$\frac{\alpha_x x + \beta_x x'}{\sqrt{\beta_x}} \longrightarrow \sqrt{\beta_x}\, x'$$

Charged Particle Moving in a Magnetic Field B

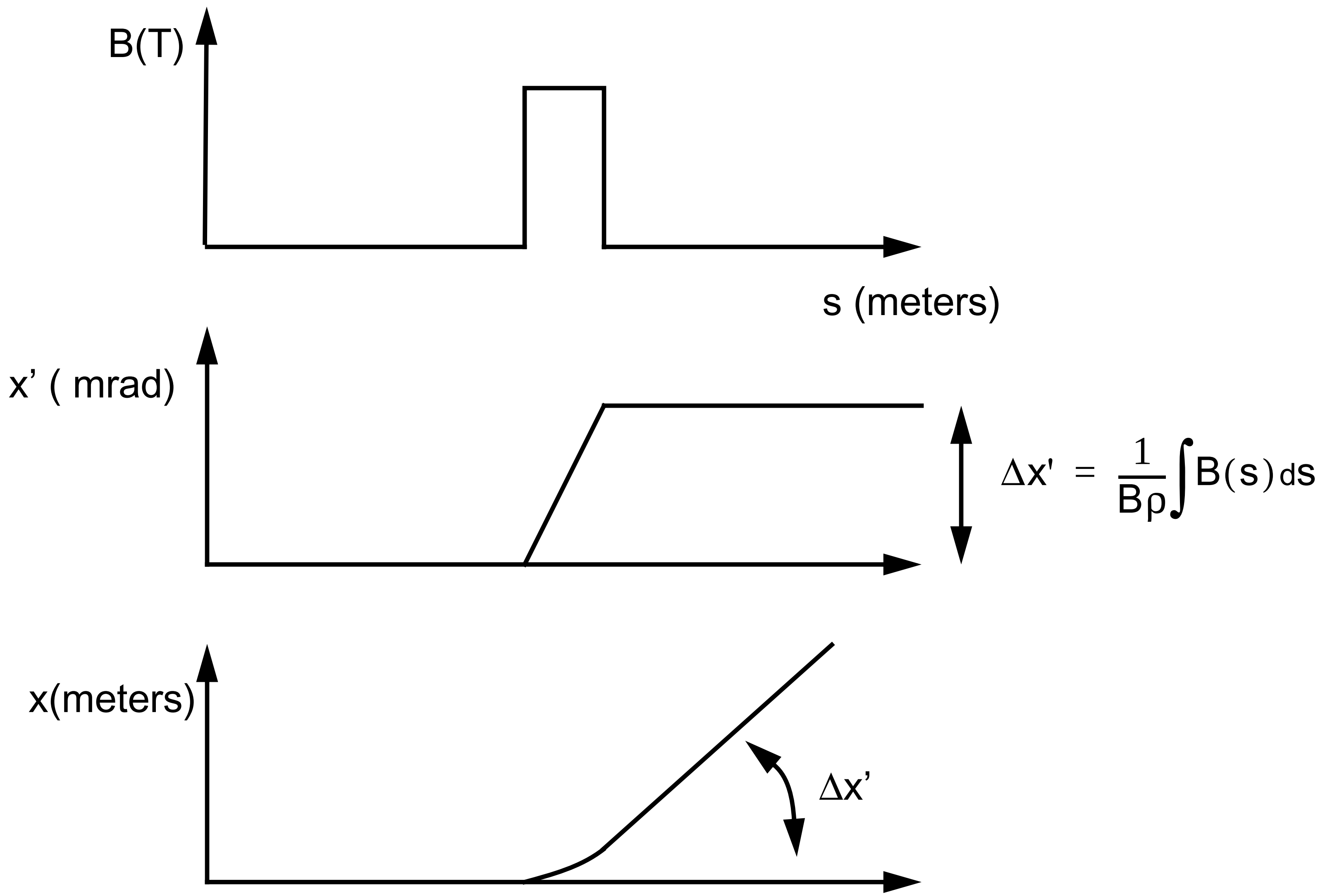


B(T) ρ (m) = 3.335641 E (GeV)

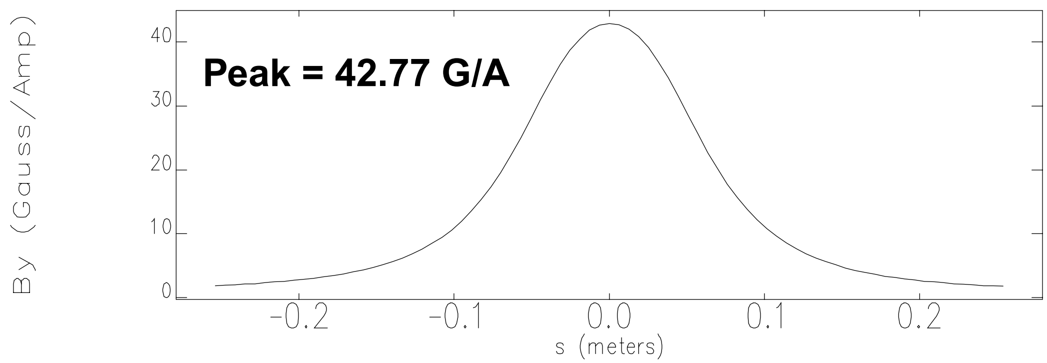
(magnetic field pointing into page)

Hard-edge dipole steering corrector

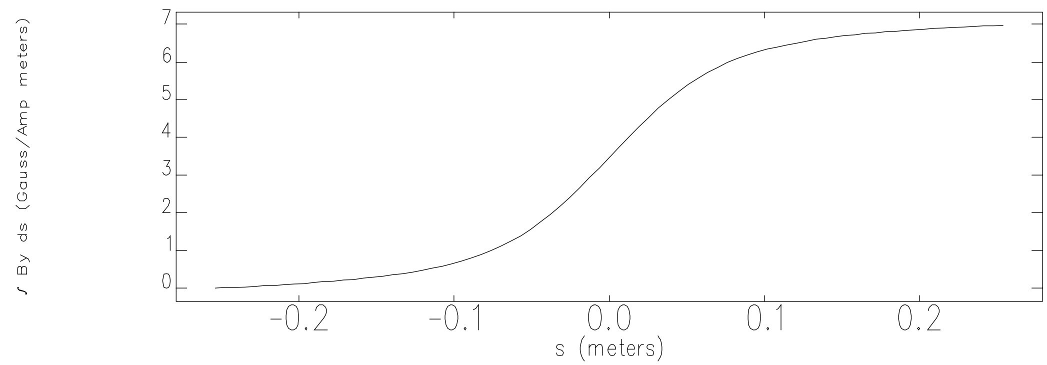
First and Second Magnetic Field Integrals, in Sensible Units



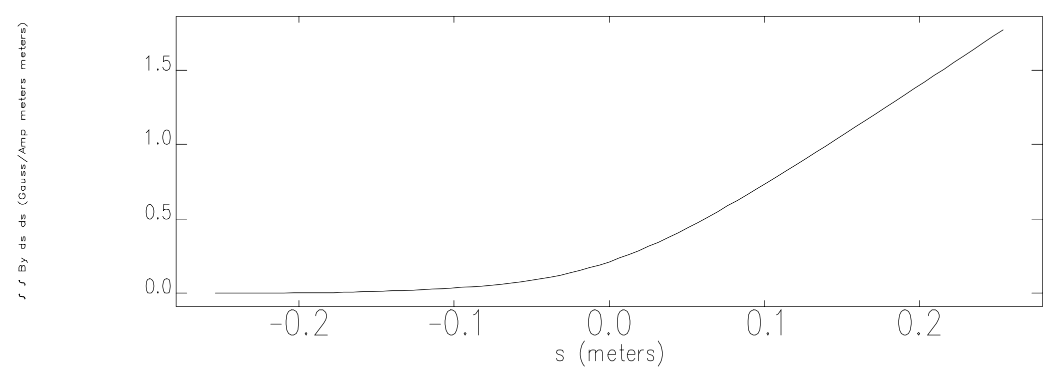
Actual Magnet Measurement Data, APS Linac Steering Corrector



Gauss / Amp



Gauss-meters / Amp

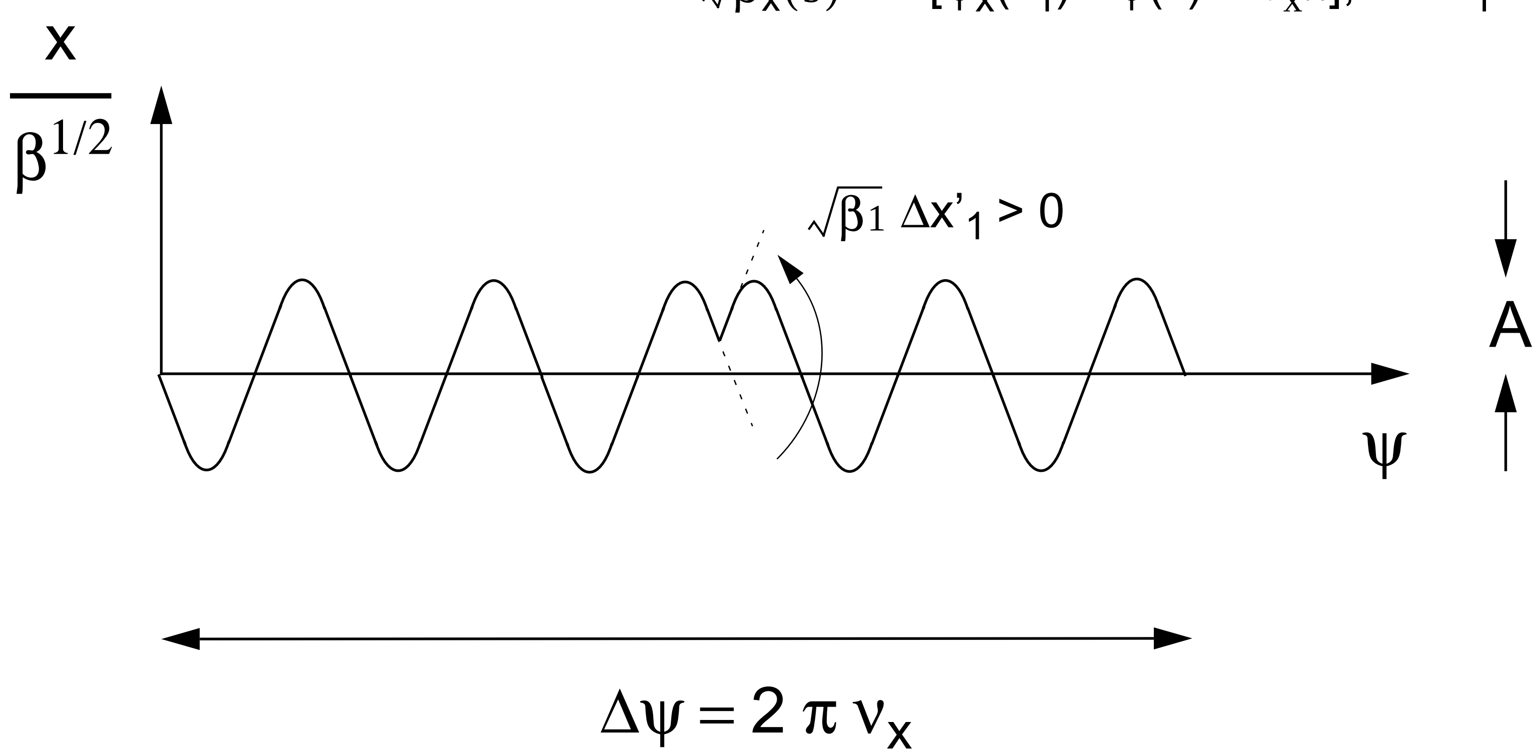


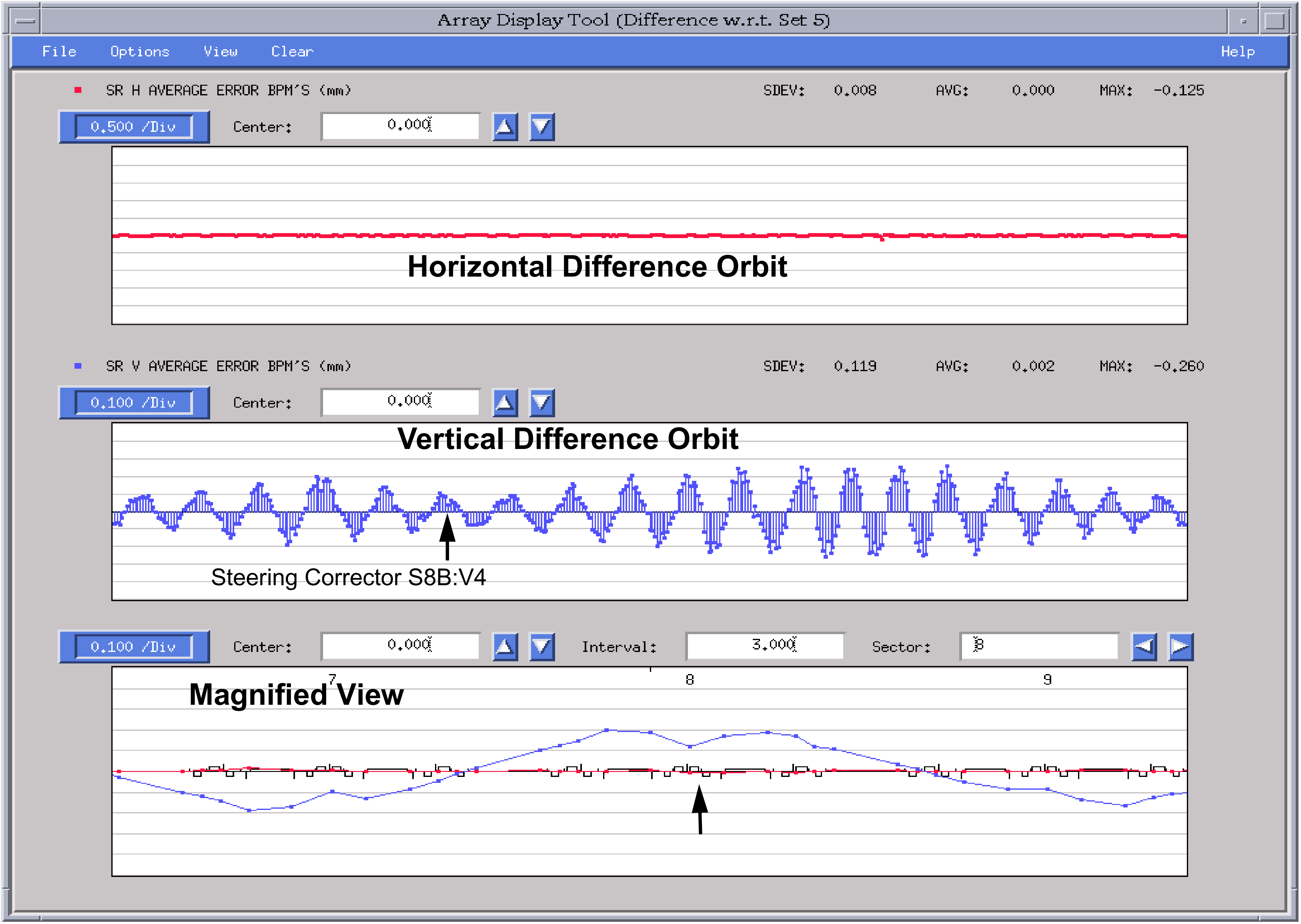
Gauss-meters^2 / Amp

Closed Orbit Distortion Resulting from Single Corrector Change $\Delta x'_1$

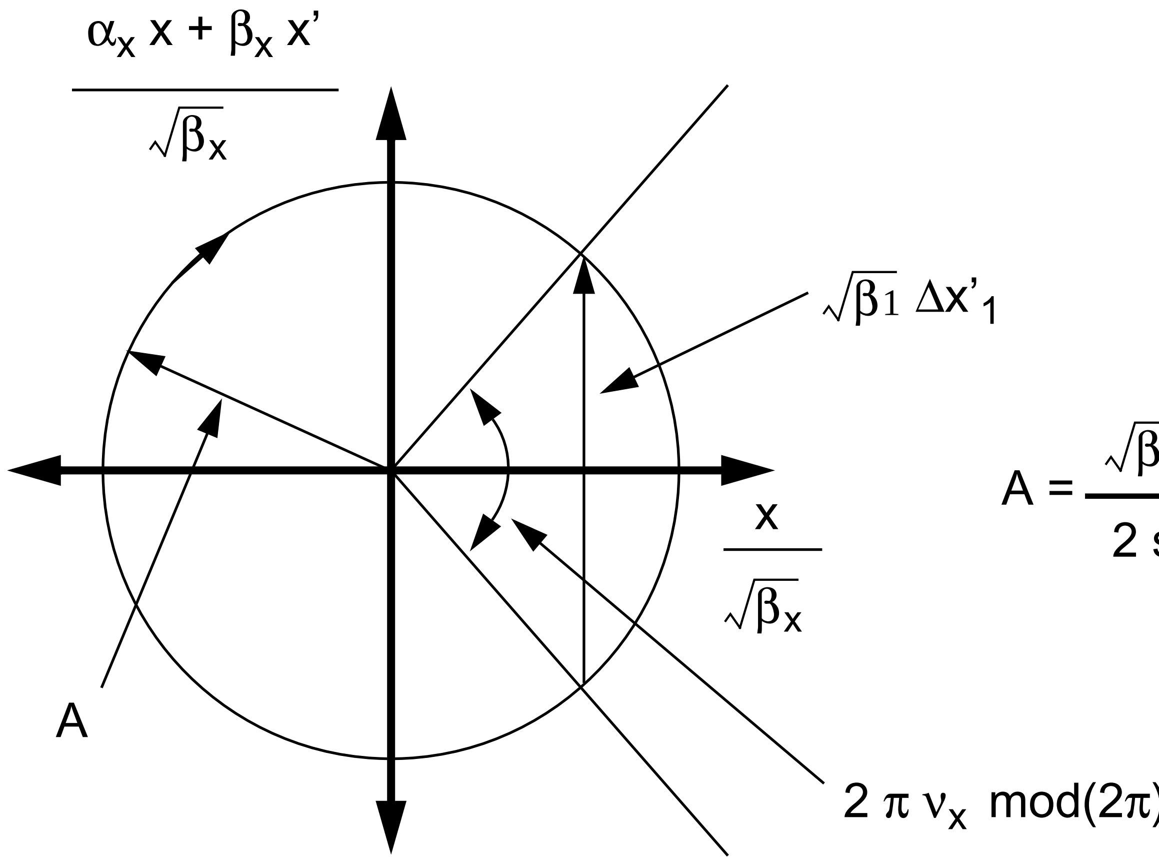
$$x(s) = A\sqrt{\beta_x(s)}\cos[\psi_x(s) - \psi(s_1) + \nu_x\pi]\ ;\ s < s_1$$

$$= A\sqrt{\beta_x(s)}\cos[\psi_x(s_1) - \psi(s) + \nu_x\pi];\ s > s_1$$




Array Display Tool (Difference w.r.t. Set 5)
File
Options
View
Clear
Help
SR H AVERAGE ERROR BPM'S (mm)
SDEV: 0.008
AVG: 0.000
MAX: -0.125
0.500 /Div
Center: 0.000
Horizontal Difference Orbit
SR V AVERAGE ERROR BPM'S (mm)
SDEV: 0.119
AVG: 0.002
MAX: -0.260
0.100 /Div
Center: 0.000
Vertical Difference Orbit
Steering Corrector S8B:V4
0.100 /Div
Center: 0.000
Interval: 3.000
Sector: 8
Magnified View
7
8
9

Closed Orbit Distortion Resulting from Single Corrector Change $\Delta x'_1$ Phase Space Representation



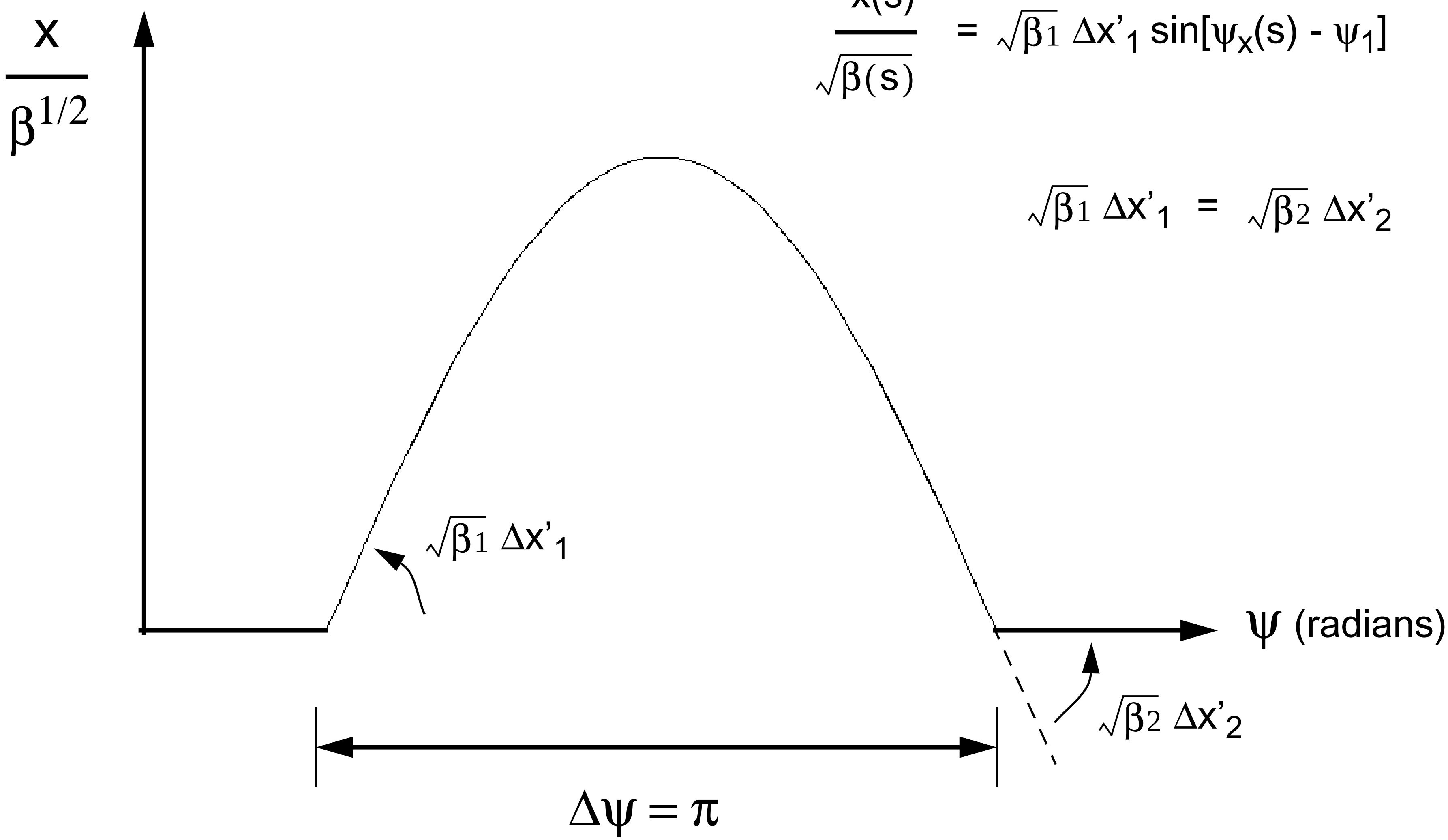
$$A = \frac{\sqrt{\beta_1}\ \Delta x'_1}{2 \sin\pi\nu_x}$$

$$x(s) = \frac{\sqrt{\beta_x(s)\beta_1}\,\Delta x_1'}{2\sin\pi\nu_x}\cos[\psi_x(s) - \psi_x(s_1) + \nu_x\pi] \quad s < s_1 \text{ etc.}$$

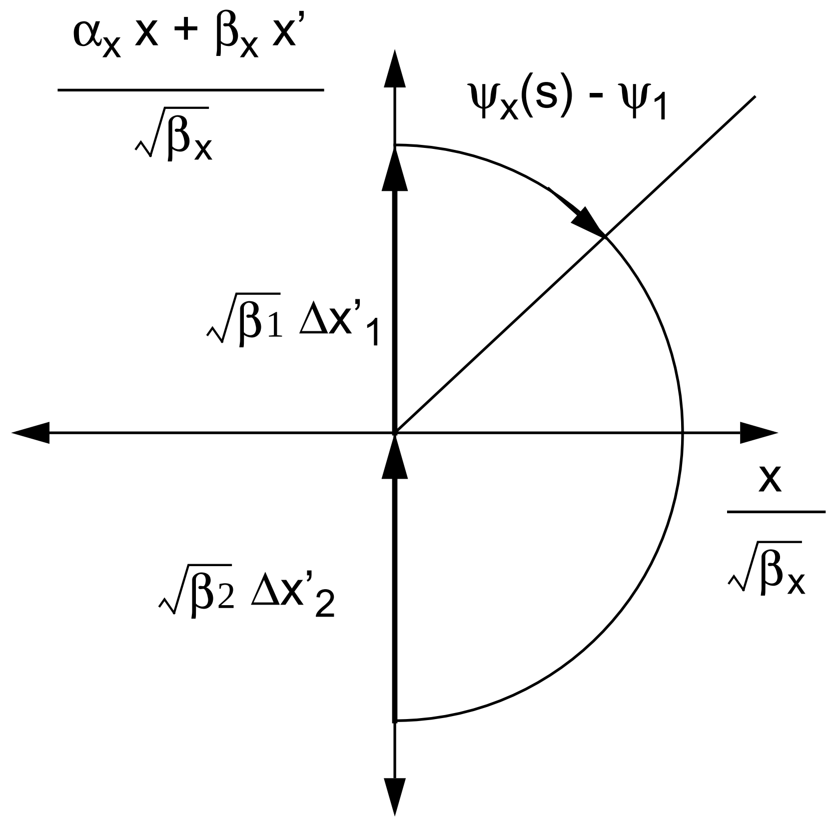
Closed Orbit Half-Wave Bump - Two Correctors

$$\frac{x(s)}{\sqrt{\beta(s)}} = \sqrt{\beta_1}\, \Delta x'_1 \sin[\psi_x(s) - \psi_1]$$

$$\sqrt{\beta_1}\, \Delta x'_1 = \sqrt{\beta_2}\, \Delta x'_2$$

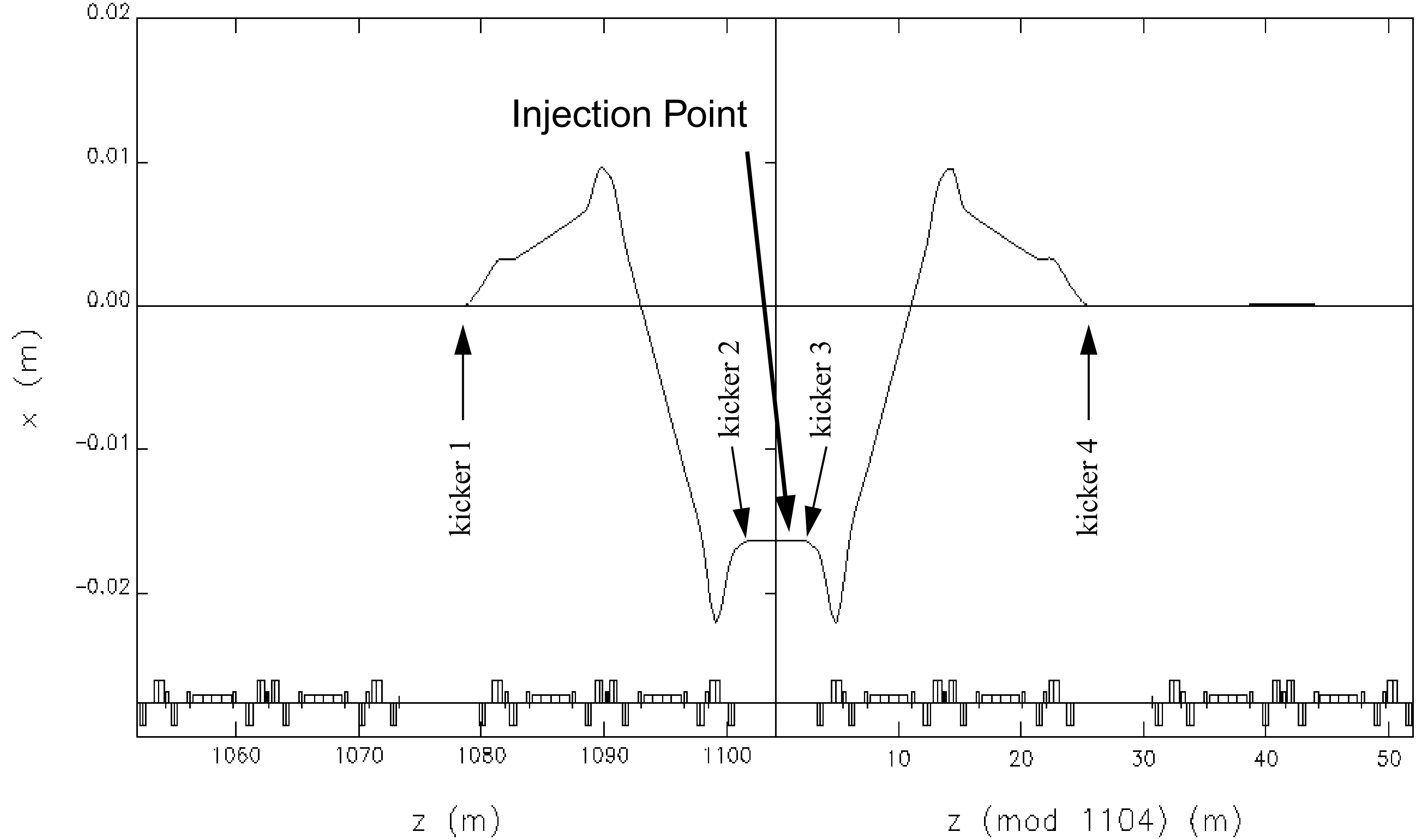


Phase Space Representation of Two-Corrector Half-Wave Bump



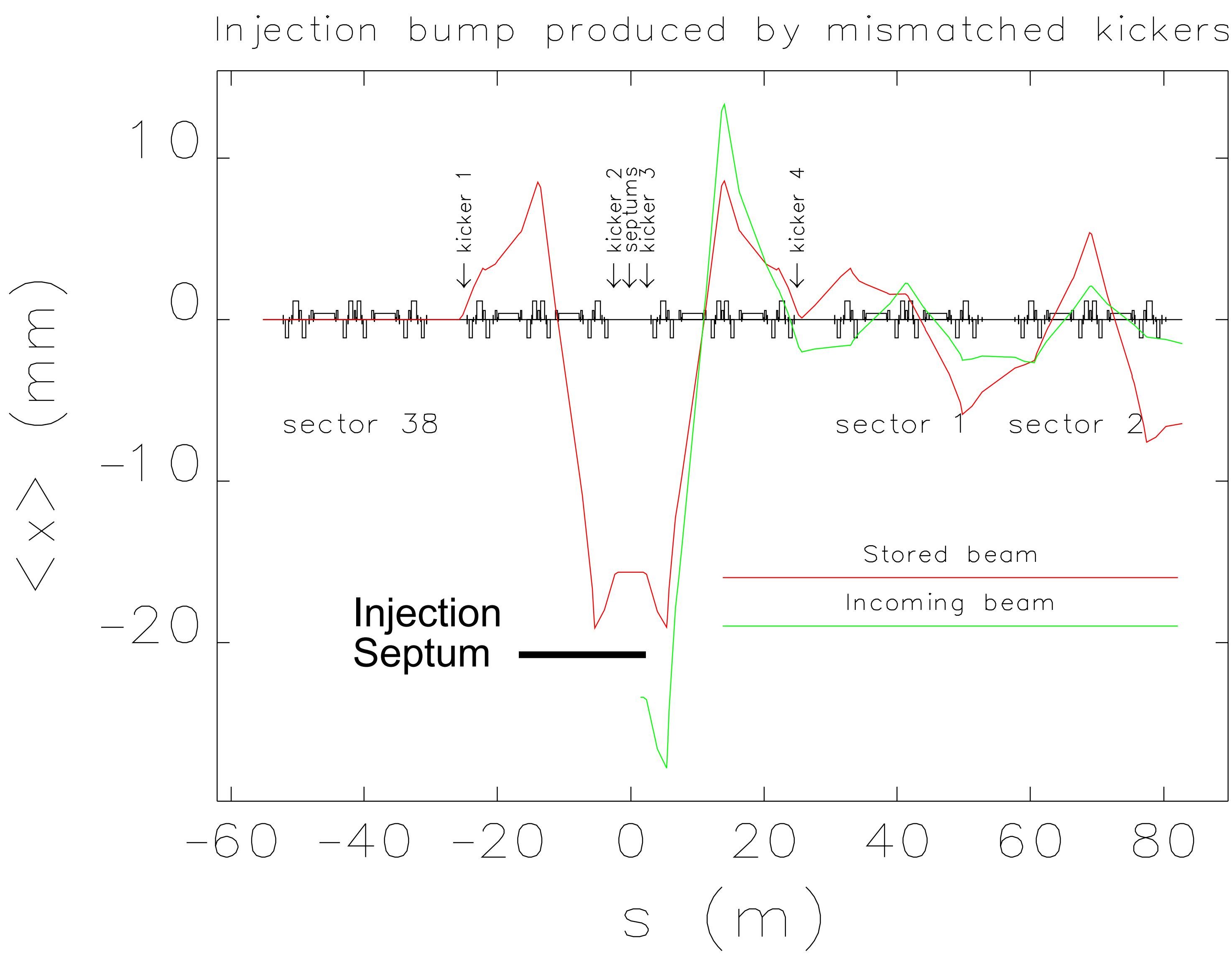
$$x_{max} = \sqrt{\beta_0 \beta_1}\, \Delta x'_1$$

APS Injection Closed Orbit Bump as Designed
(Pulsed - Full Width < 2 turns = 7 μsec)



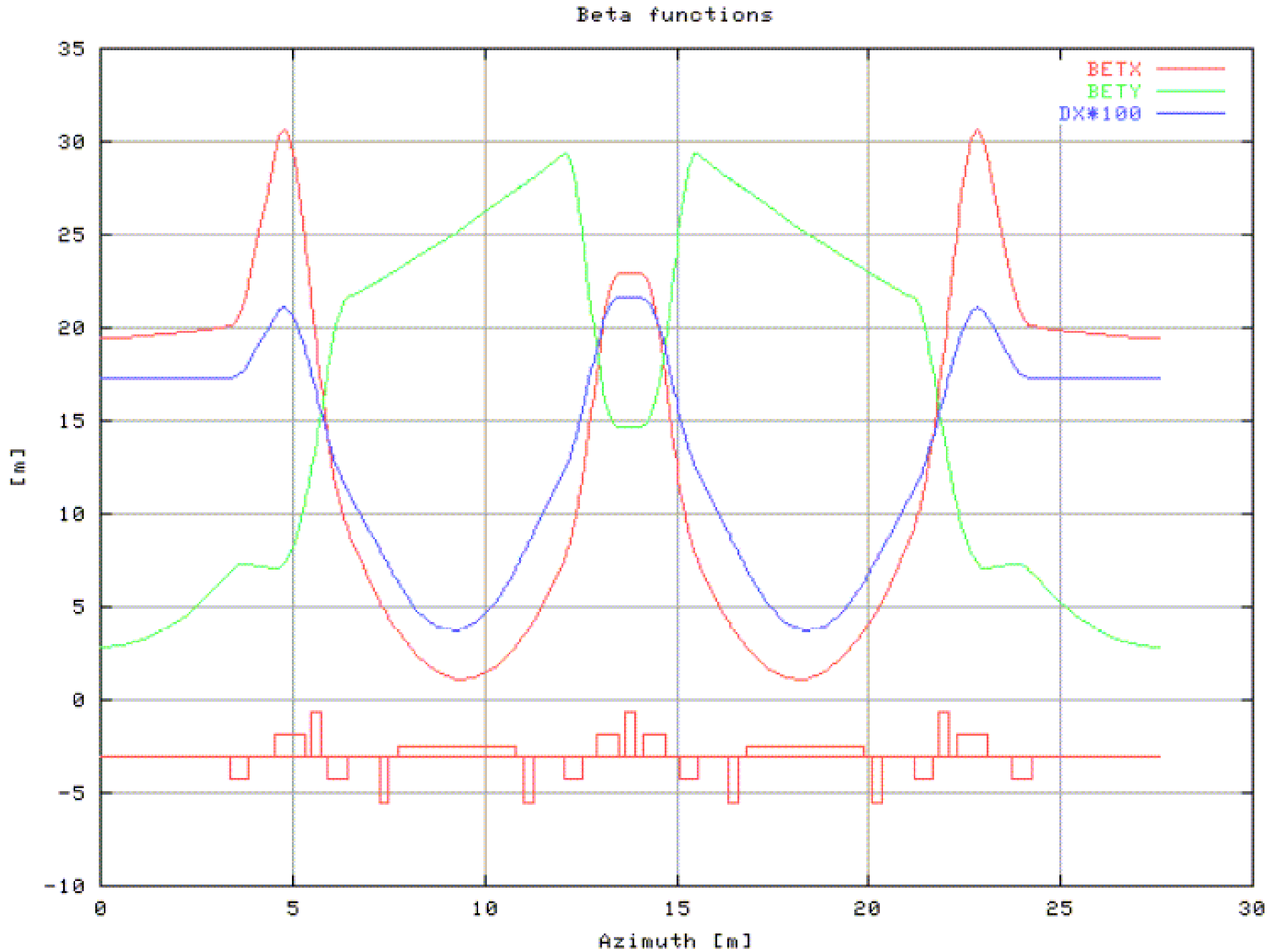
Maximum orbit bump during injection

APS Injection Orbit Bump in Practice (not closed)



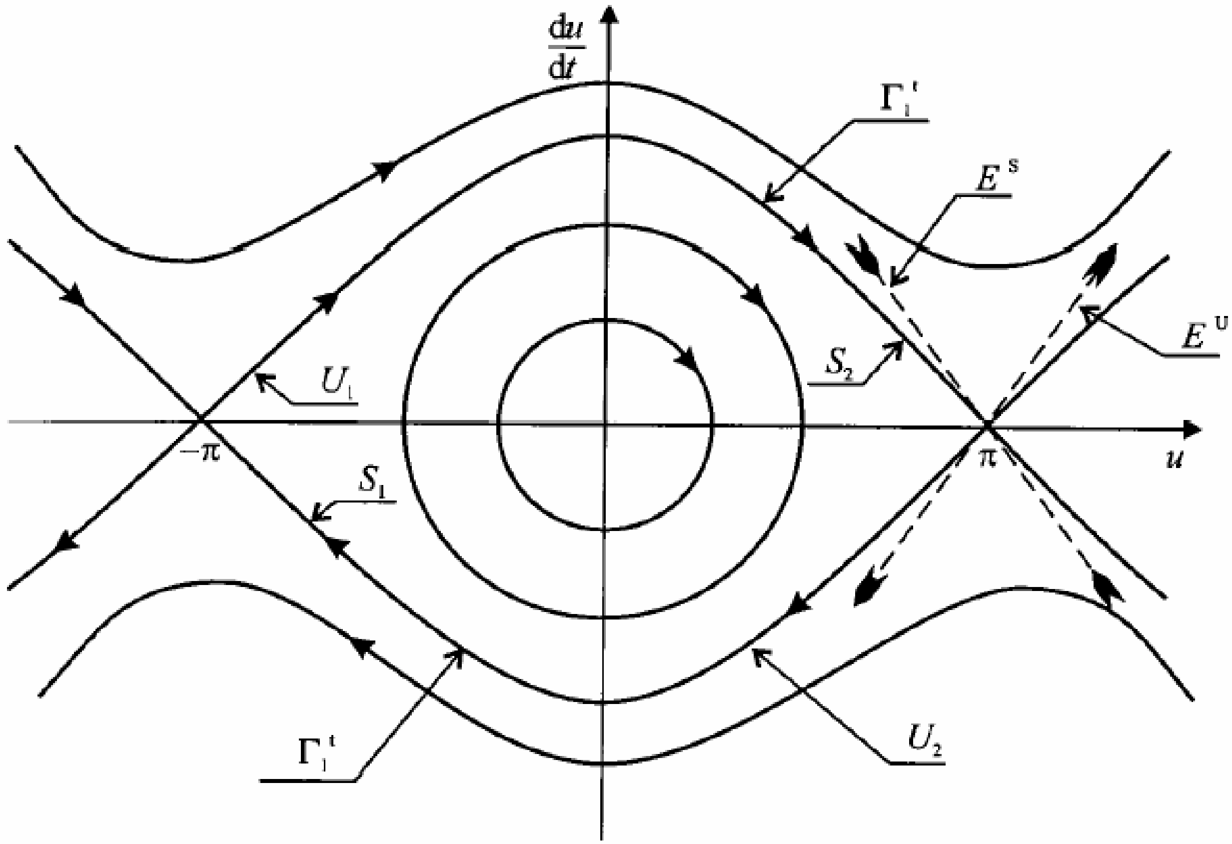
APS lattice functions

2.4 nm×rad lattice, one sector



V. Sajaev APS

Essentials of Accelerator Physics
Longitudinal Phase Space



Phase-space portrait of a pendulum

Kapitaniak, T. Chaos for engineers : theory, applications, and control. Springer Verlag, 1998

Phase Stability

The longitudinal equations of motion for an electron in a storage ring are

$$\frac{d\phi}{dt} = -\alpha\,\omega_{RF}\,\varepsilon \qquad \frac{d\delta}{dt} = \frac{eV_{RF}(t) - U(\delta)}{E_0 T_0} = \frac{eV^0_{RF}}{E_0 T_0}[\sin\phi - \sin\phi_s]$$

ϕ = Phase of arrival at a fixed point along the closed orbit, in radians, at the RF frequency.
$= \omega_{RF}\,\Delta t$

$\omega_{RF} = 2\,\pi\,f_{RF}$ = Angular RF frequency

α_c = momentum compaction

$$= \frac{(\Delta L)/L}{(\Delta E)/E}$$

$T_0 = \dfrac{1}{f_{rev}}$ = Revolution Period

δ = relative energy deviation = $\dfrac{\Delta E}{E_0}$

E_0 = Nominal ring energy

$$V_{RF}(t) = eV^0_{RF}\sin(\omega_{RF}\Delta t) = eV^0_{RF}\sin(\phi)$$

= RF voltage gain per turn

ϕ_s = Synchronous phase
- defined by the relation for $U(\delta)$:

$$U(\delta) = eV^0_{RF}\sin\phi_s$$

= Energy loss per turn from emission of synchrotron radiation and other parasitic losses

Define Phase ψ relative to synchronous phase: $\psi \equiv (\phi - \phi_s)$

Taylor expand $[\sin\phi - \sin\phi_s] = (\phi - \phi_s)\cos\phi_s + \text{order}(\phi - \phi_s)^3 = \psi\cos\phi_s$

Pendulum Equation

$$\frac{d^2\phi}{dt^2} = -\frac{\alpha_c \omega_{RF} e V^0_{RF}}{E_0 T_0}[\sin\phi - \sin\phi_s] = \left(-\frac{\alpha_c \omega_{RF} e V^0_{RF} \cos\phi_s}{E_0 T_0}\right)(\phi - \phi_s)$$

$$\longrightarrow \quad \frac{d^2\psi}{dt^2} + \Omega_s^2\,\psi = 0$$

A harmonic oscillator equation

Ω_s = Angular synchrotron frequency $2\pi f_s$

$$= \sqrt{\frac{\alpha_c \omega_{RF} e V^0_{RF} \cos\phi_s}{E_0 T_0}}$$

$$\frac{d^2\psi}{dt^2} + \Omega_s^2\,\psi = 0 \qquad \Omega_s = \sqrt{\frac{\alpha_c \omega_{RF} e V_{RF}^0 \cos\phi_s}{E_0 T_0}}$$

$$\frac{\Omega_s}{\omega_{rev}} = \frac{\Omega_s T_0}{2\pi} = \nu_s$$ = Synchrotron Tune = Number of synchrotron oscillations / turn

Arrival phase

$$\psi = \psi_0 \cos\Omega_s t$$

Relative energy deviation

$$\delta = \frac{\Delta E}{E_0} = -\frac{1}{\alpha\omega_{RF}}\frac{d\psi}{dt} = \delta_{max}\sin\Omega_s t$$

$$\delta_{max} = \frac{\Omega_s}{\alpha\omega_{RF}}\psi_0 = \frac{\Omega_s}{\alpha h \omega_{rev}}\psi_0 = \frac{\nu_s}{\alpha h}\psi_0$$

$$h = \frac{\omega_{RF}}{\omega_{rev}}$$ = ring harmonic number

(usually a big number; = 1296 for APS)

Harmonic Oscillator

$$\psi = \psi_0 \cos \Omega_s t \qquad \delta = \delta_{max} \sin \Omega_s t = \frac{\nu_s}{\alpha_c h} \psi_0 \sin \Omega_s t$$

“Energy” of a harmonic oscillator

$$\frac{1}{2} k x^2 + \frac{1}{2} m v^2 \longrightarrow$$

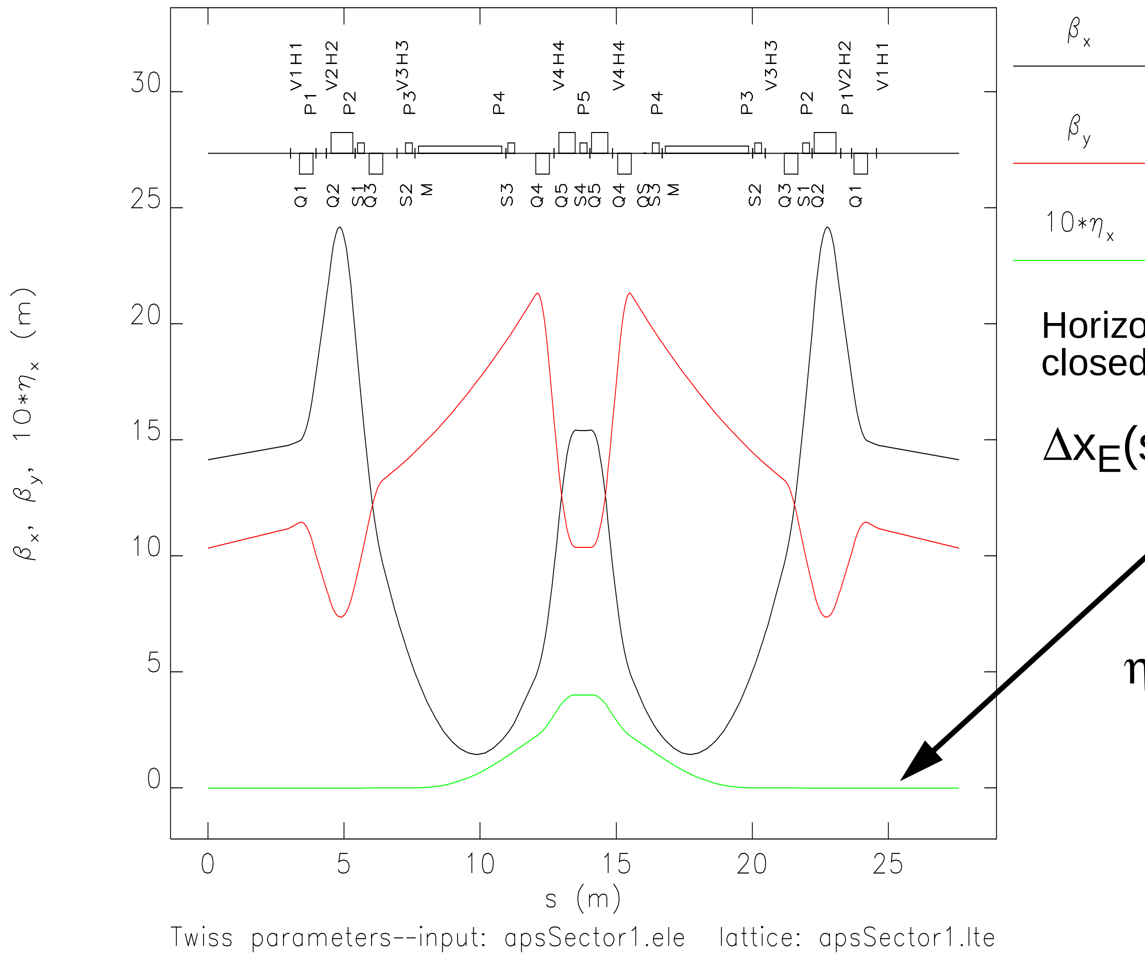
“Energy” of a single particle’s synchrotron oscillation

$$\left(\alpha_c{}^2 h^2\right)\delta^2 + \left(\nu_s^2\right)\psi^2 = \text{constant}$$

Important Longitudinal Phase Space Parameters

Quantity	Definition	Units	Typical Values
$\delta = (\Delta E / E)$	Relative energy deviation variable	Dimensionless	Fractions of a percent
$\psi = \omega_{RF} \Delta\tau$	Arrival phase variable	Dimensionless	A few degrees
$\alpha_c = \frac{(\Delta L)/L}{(\Delta E)/E}$	Momentum Compaction	Dimensionless	Big rings e.g. APS - A few * 10^{-4} Med. rings e.g. ALS - A few * 10^{-3} Small rings e.g. NSLS VUV - A few * 10^{-2}
$h = \omega_{RF} / \omega_{rev}$	Harmonic number	Dimensionless	1 to 1000's
$\nu_s = \Omega_s / \omega_{rev}$	Synchrotron tune	Dimensionless	a few * 10^{-2}
$f_{RF} = \omega_{RF} / 2\pi$	RF frequency	Hz	Rings 10's to 100's of MHz, SRF 1.3 GHz e.g. ERL's Electron Linacs 3-30 GHz
$f_{rev} = \omega_{rev} / 2\pi$	Revolution frequency	Hz	3e8 / Circumference
$U(\delta) = eV^0_{RF} \sin\phi_s$	Energy Loss per Turn	keV - MeV	$= C_\gamma E^4 / \rho$

Connection between longitudinal and horizontal beam motion



APS 8.2 nm-rad

Horizontal displacement of the equilibrium closed orbit for an off-energy particle:

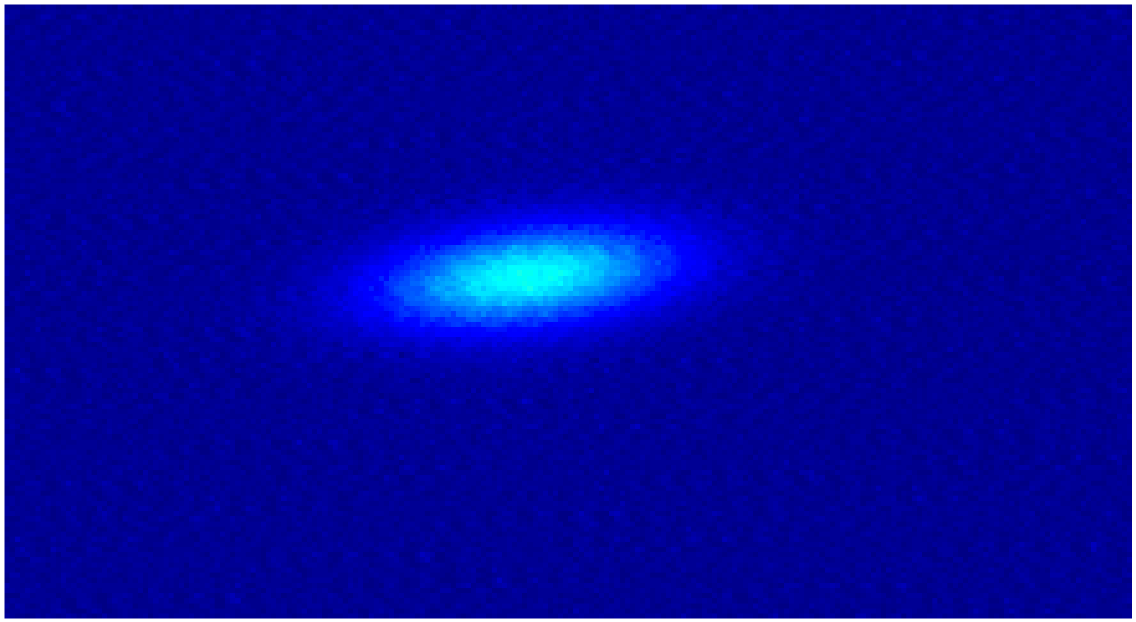
$$\Delta x_E(s) = \eta(s)\,\frac{\Delta E}{E_0} = \eta(s)\,\delta$$

$\eta(s)$ = the dispersion

Particle Beam Statistical Properties

Beam Current (Amps) = $N_{e-} * 1.6e\text{-}19$ (Coulombs / electron) $* f_{rev}$ (Hz)

N_{e-} = Number of electrons = a few $* 10^{12}$



Summary	X Fit	Y Fit
Centroid	−104.83	−117.99
Sigma	461.79	393.24
FWHM	1085.55	682.98
Units	um	um

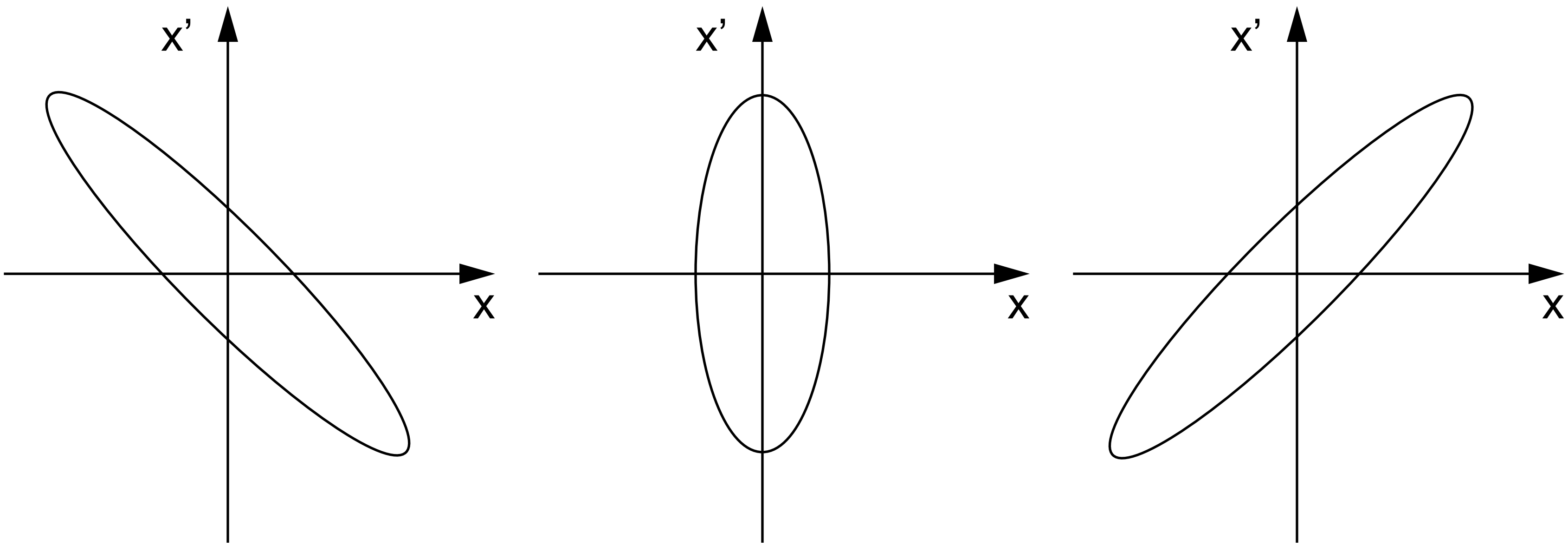
Particle Beams with > 1 particle

Single particle trajectory $x(s) = [W_x\, \beta_x(s)]^{1/2} \cos[\psi_x(s) - \psi_{0x}]$

Courant-Snyder Invariant* $W_x = \gamma_x(s)\, x^2(s) + 2\, \alpha_x(s)\, x(s)\, x'(s) + \beta_x(s)\, x'(s)^2$

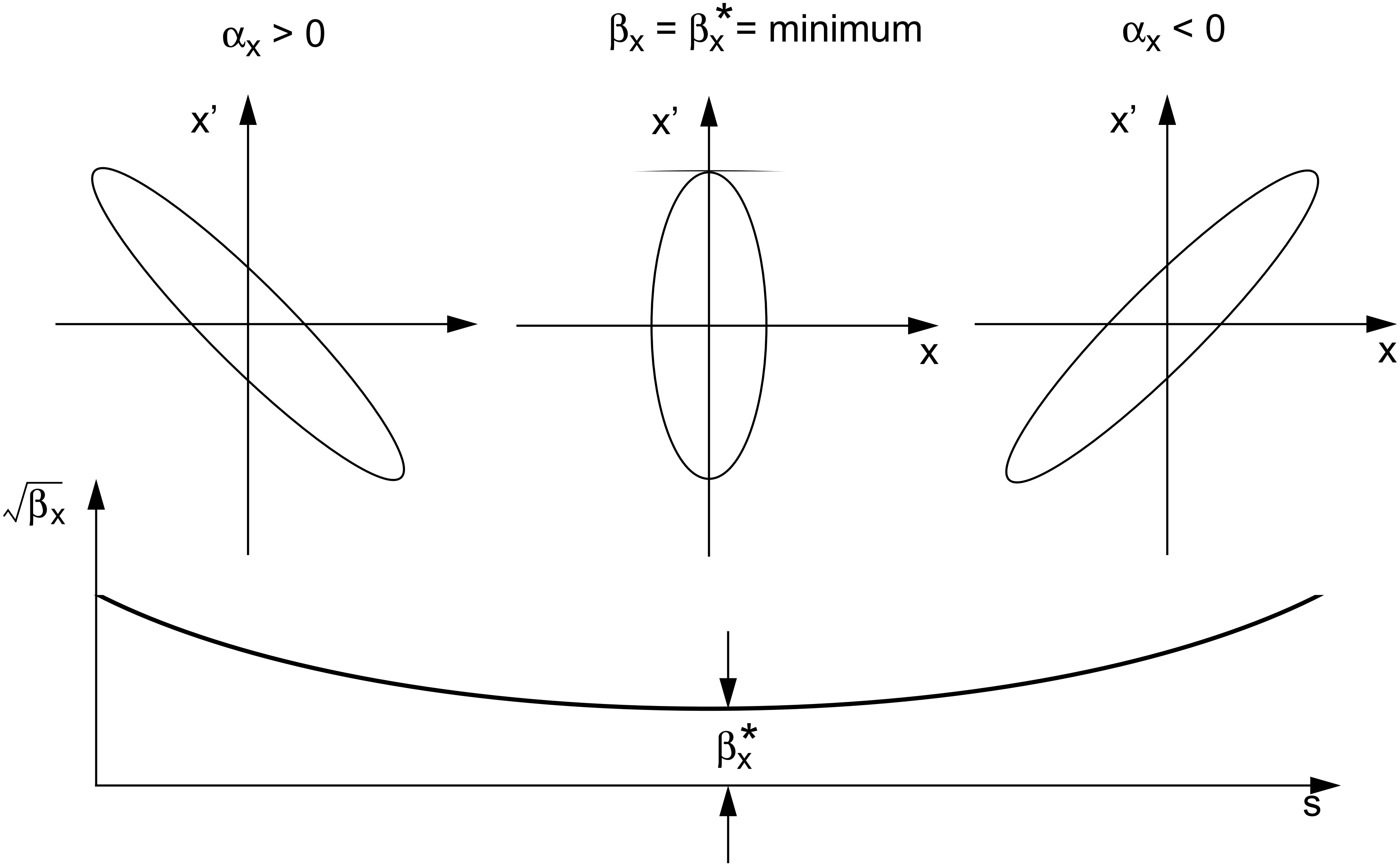
$= [\, x^2 + (\, \alpha_x\, x + \beta_x\, x')^2\,] / \beta_x$

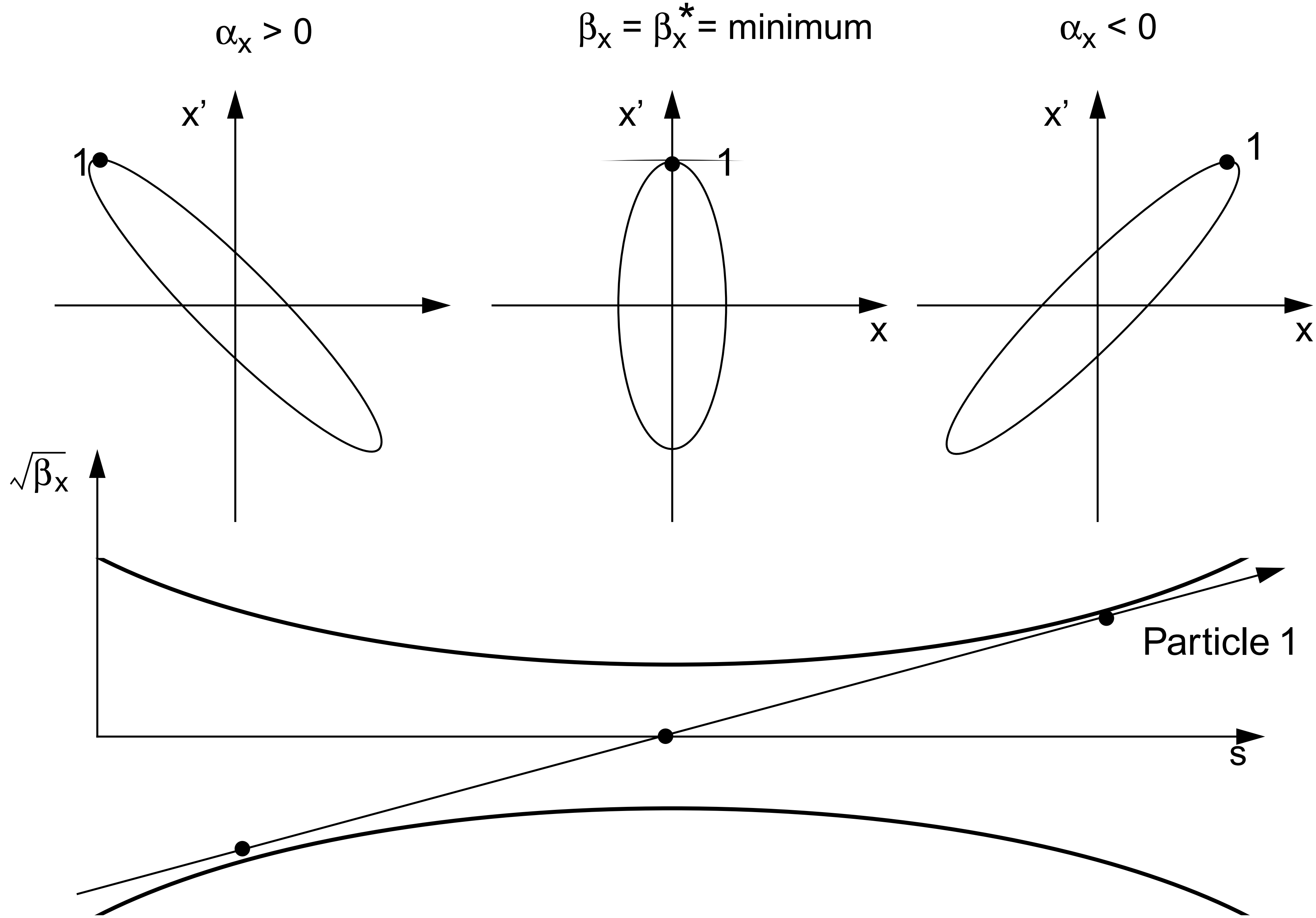
... and similarly for x <--> y



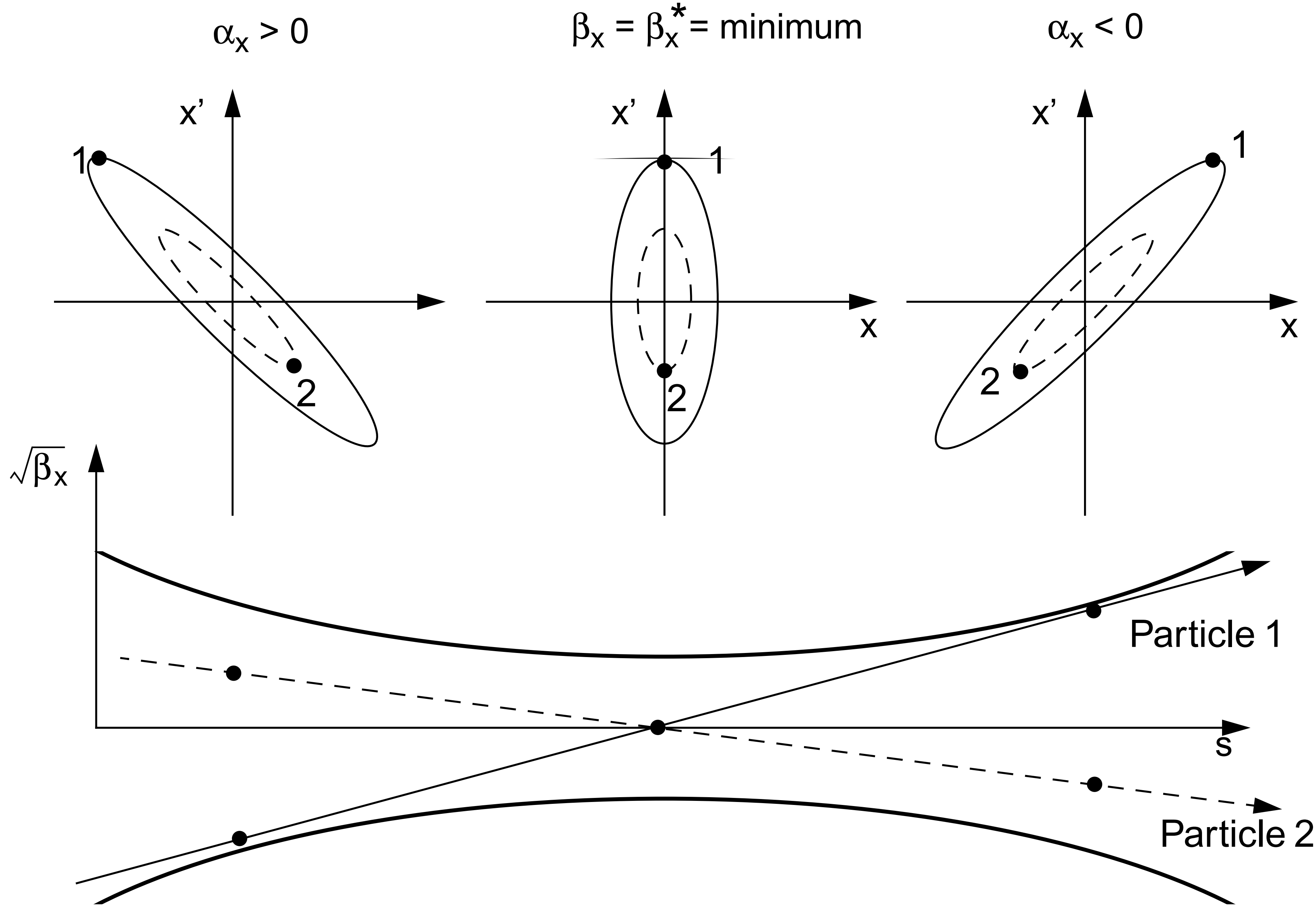
$\alpha_x > 0$ β_x = minimum $\alpha_x < 0$

Propagation of Phase Space Through Drift Space

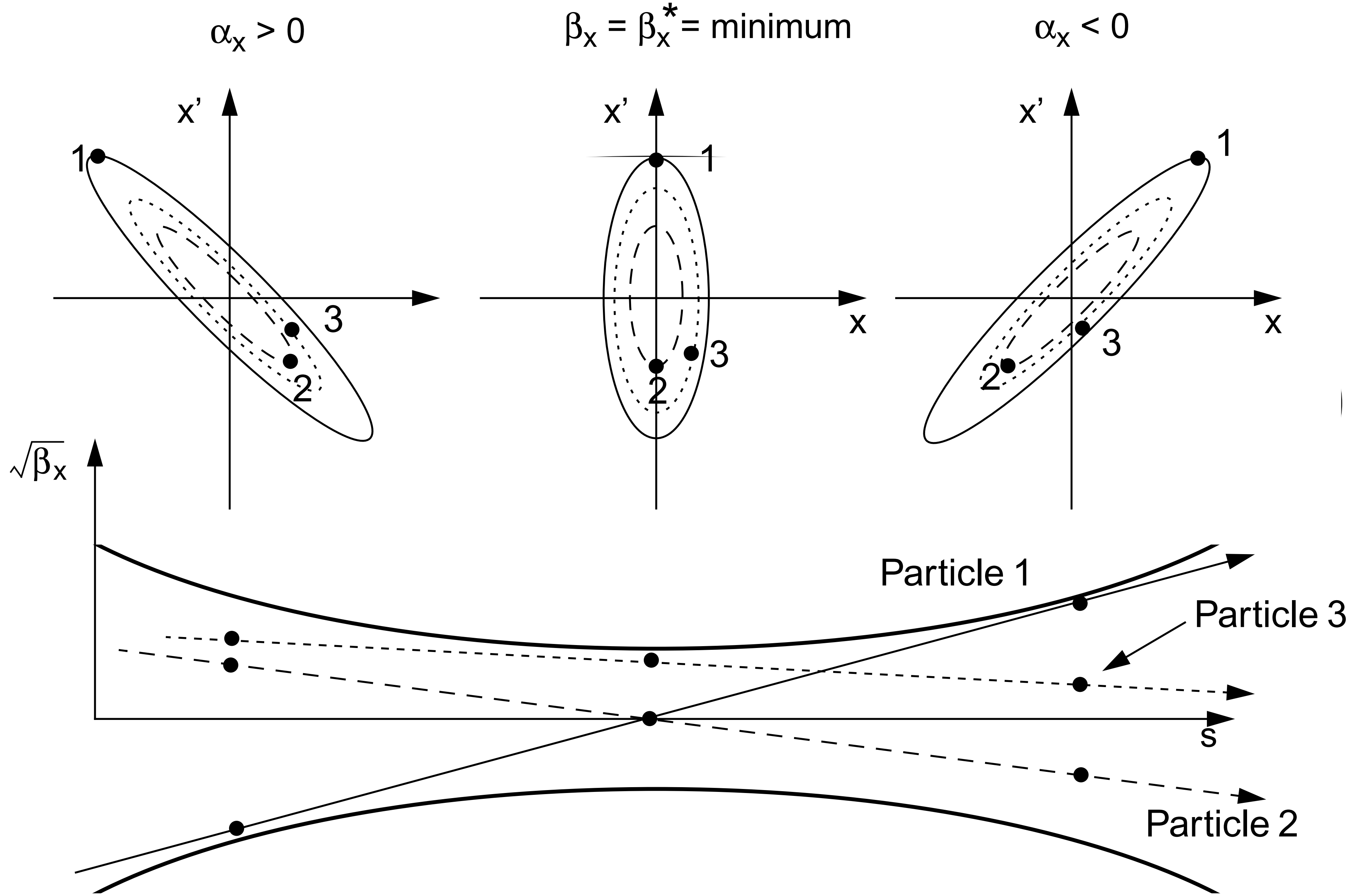




$\alpha_x > 0$
$\beta_x = \beta_x^* =$ minimum
$\alpha_x < 0$
x'
x
1
$\sqrt{\beta_x}$
Particle 1
s

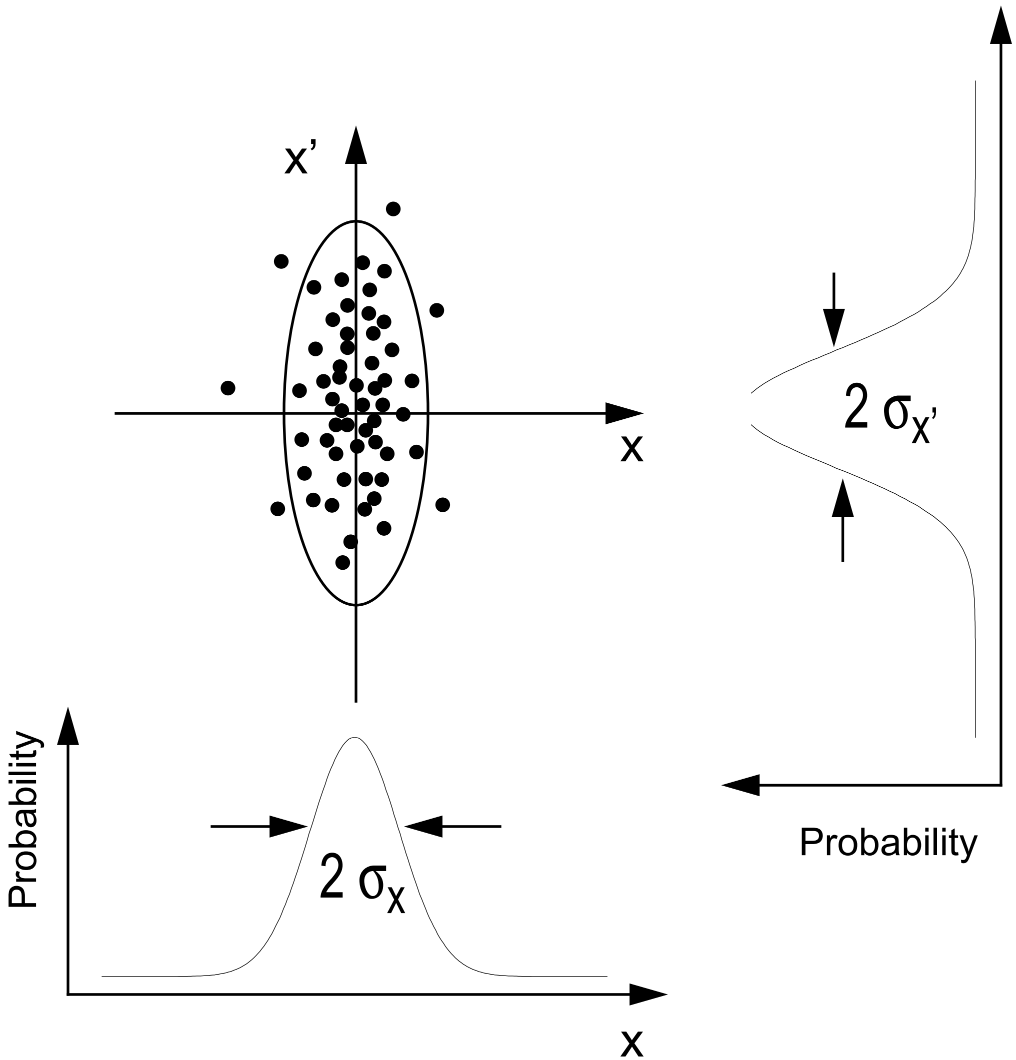


$\alpha_x > 0$
$\beta_x = \beta_x^* =$ minimum
$\alpha_x < 0$
x'
x
1
2
$\sqrt{\beta_x}$
s
Particle 1
Particle 2



$\alpha_x > 0$
$\beta_x = \beta_x^* =$ minimum
$\alpha_x < 0$
x'
x
1
2
3
$\sqrt{\beta_x}$
s
Particle 1
Particle 3
Particle 2

Emittance



$$\sigma_x = \sqrt{\beta_x \varepsilon_x}$$

ε_x = Horizontal Emittance

$= \sigma_x \sigma_{x'}$

(when $\alpha_x = 0$)

Beam Size and Angular Divergence

In the horizontal plane, there is an additional contribution to beam size and angular divergence derived from the finite fractional energy spread σ_δ .

A single off-energy particle will be radially displaced relative to the equilibrium closed orbit by an amount

$$\Delta x_E(s) = \eta(s)\,\frac{\Delta E}{E_0} = \eta(s)\,\delta$$

where E_0 is the nominal beam energy, ΔE is the energy error, and $\eta(s)$ is the horizontal dispersion function.

The fractional energy spread σ_δ makes a contribution to the horizontal beam size of $\eta(s)\,\sigma_\delta$. This adds in quadrature to $\sqrt{\beta_x \varepsilon_x}$. A similar contribution is made to the horizontal angular divergence $\sigma_{x'}$. For planar machines, the vertical plane is unaffected by energy spread.

$$\sigma_y = \sqrt{\beta_y \varepsilon_y}$$

$$\sigma_{y'} = \sqrt{\gamma_y \varepsilon_y} = \frac{\sigma_y}{\beta_y}$$

(generally) (when $\alpha_y = 0$)

$$\sigma_x = \sqrt{\beta_x \varepsilon_x + \eta^2 \sigma_\delta^2}$$

$$\sigma_{x'} = \sqrt{\gamma_x \varepsilon_x + \eta'^2 \sigma_\delta^2}$$

Twiss parameter relations

$$\gamma(s) = \frac{1 + \alpha^2(s)}{\beta(s)} \qquad \alpha(s) = -\,\beta'(s)\,/\,2$$

Emittance vs. Courant-Snyder Invariant

The emittance ε_y is a property of an ensemble of charged particles stored in a ring (or accelerated in a Linac):

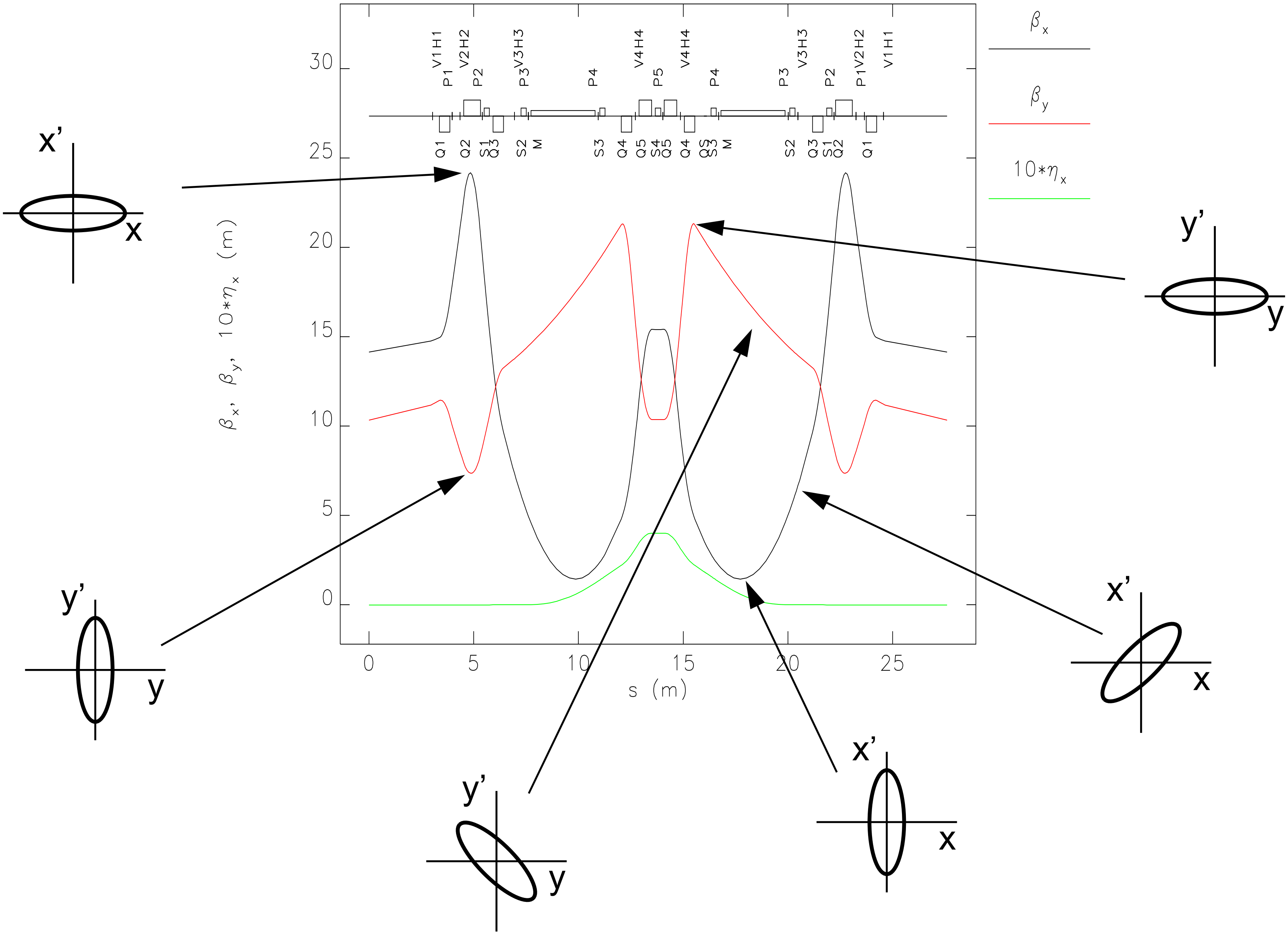
$$\sigma_y = \sqrt{\beta_y \varepsilon_y}$$

The quantity

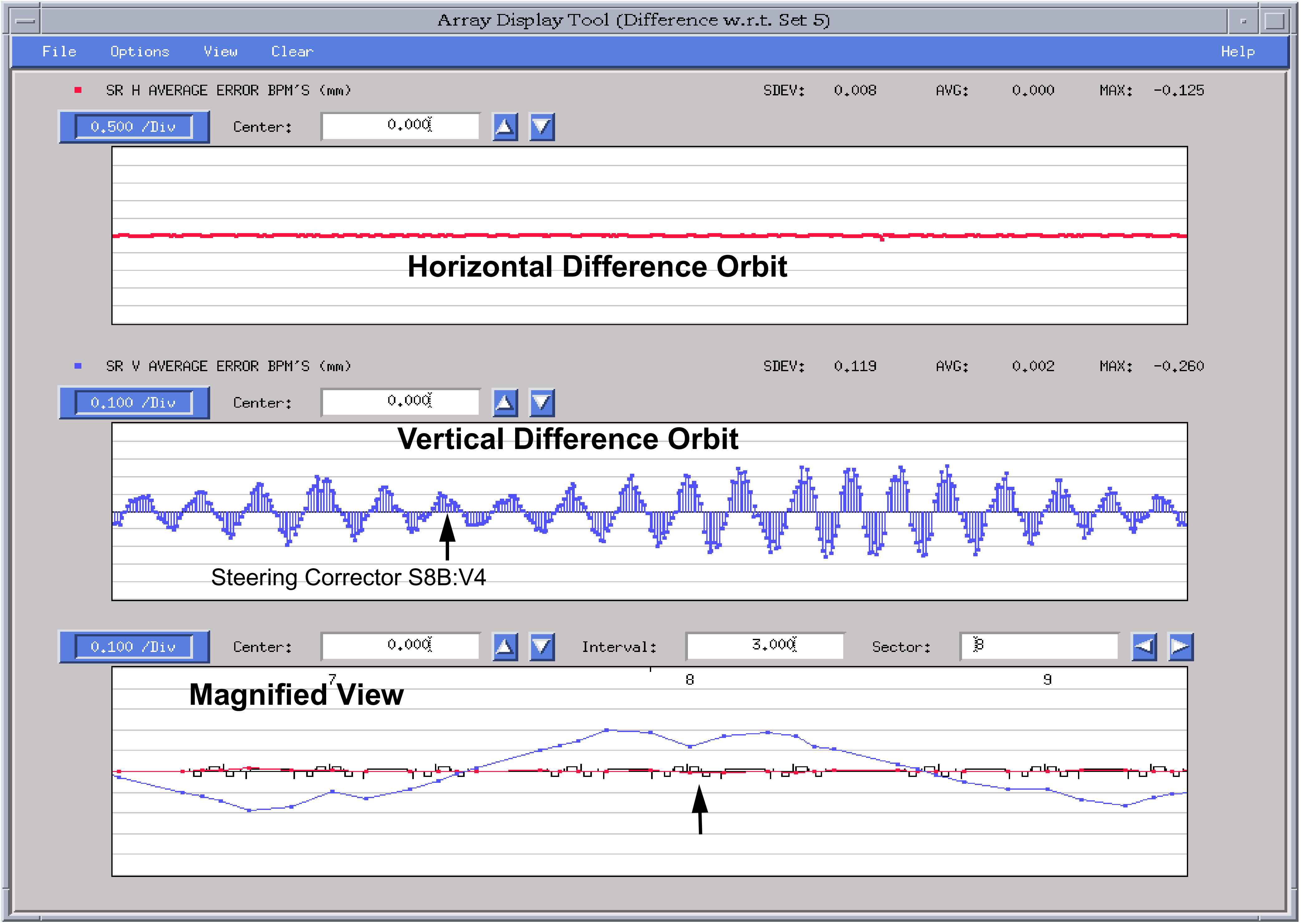
$$A = \sqrt{\beta_y W_y}$$

is the amplitude of a single particle's betatron oscillation:

$$y(s) = [W_y \, \beta_y(s)]^{1/2} \cos[\psi_y(s) - \psi_{0y}]$$



β_x
β_y
$10*\eta_x$
β_x, β_y, $10*\eta_x$ (m)
s (m)
V1H1 V2H2 V3H3 V4H4 V4H4 V3H3 P1V2H2 V1H1
P1 P2 P3 P4 P5 P4 P3 P2
Q1 Q2 S1 Q3 S2 M S3 Q4 Q5 S4 Q5 Q4 Q5 S3 M S2 Q3 S1 Q2 Q1
x'
x
y'
y


Array Display Tool (Difference w.r.t. Set 5)
File
Options
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SR H AVERAGE ERROR BPM'S (mm)
SDEV: 0.008
AVG: 0.000
MAX: -0.125
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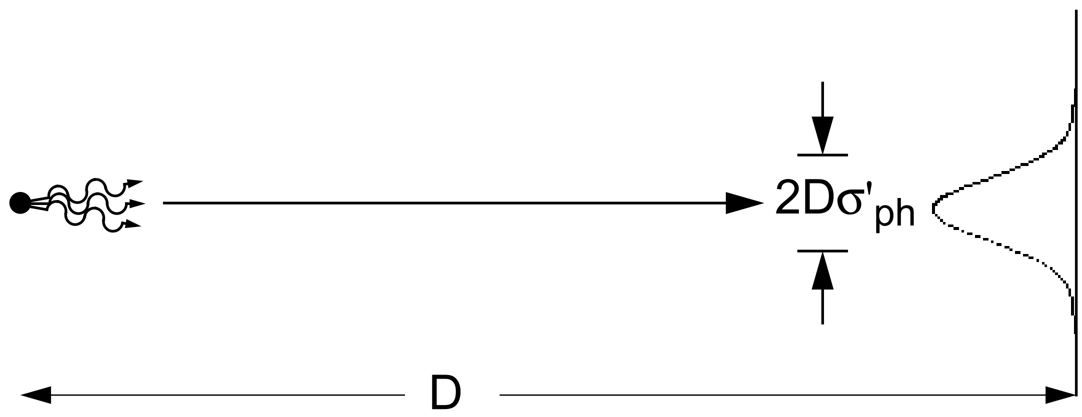
Center: 0.000
Vertical Difference Orbit
Steering Corrector S8B:V4
0.100 /Div
Center: 0.000
Interval: 3.000
Sector: 8
7
8
9
Magnified View

Photon Emittance

In the limit of a zero-emittance charged particle beam, the emitted photon beam will have a characteristic equivalent opening angle σ'_{ph} .

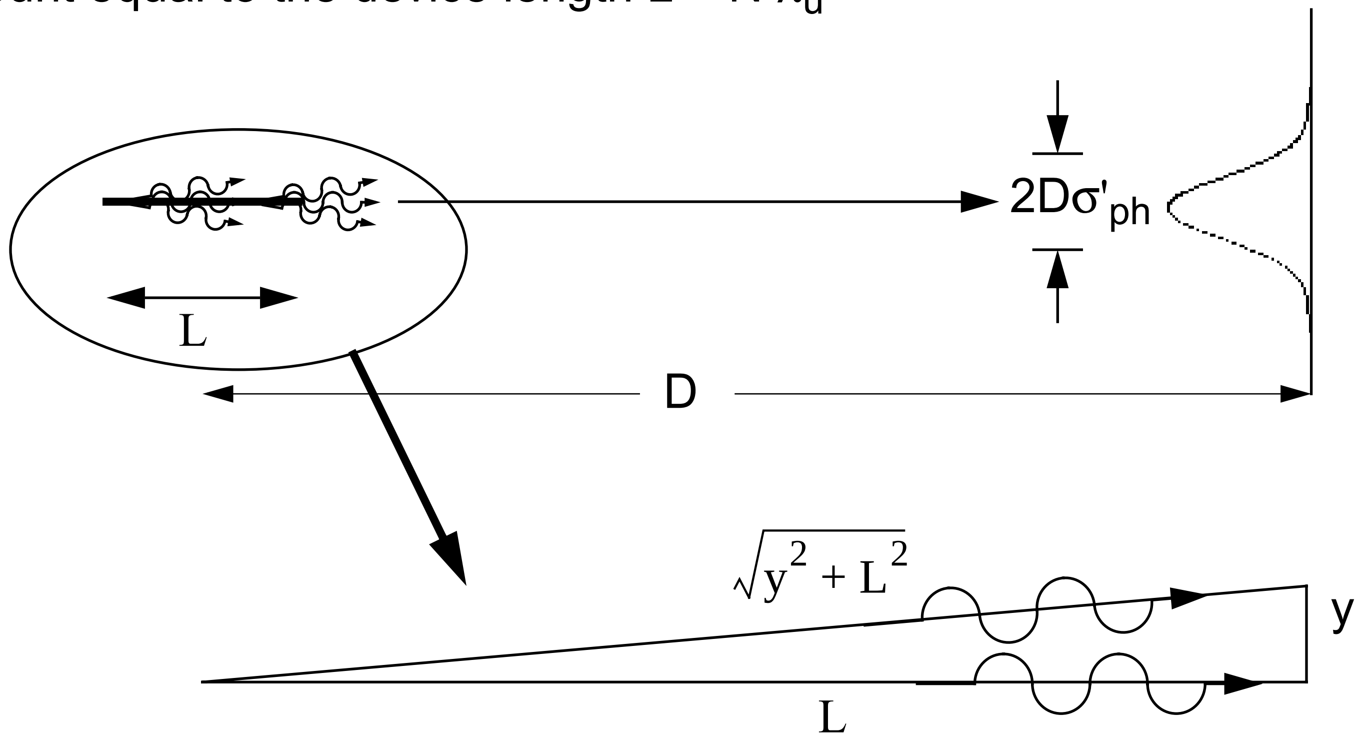


Integrated over all frequencies, σ'_{ph} is approximately equal to $1/\gamma$ (= 0.608 / γ for a bending magnet source in the vertical plane). This photon angular divergence gets added in quadrature to the electron beam angular divergence to arrive at the total photon angular divergence $\Sigma_{y'}$ for a finite emittance beam:

$$\Sigma_{y'} = \sqrt{\sigma_{y'}^2 + (\sigma_{ph}')^2}$$

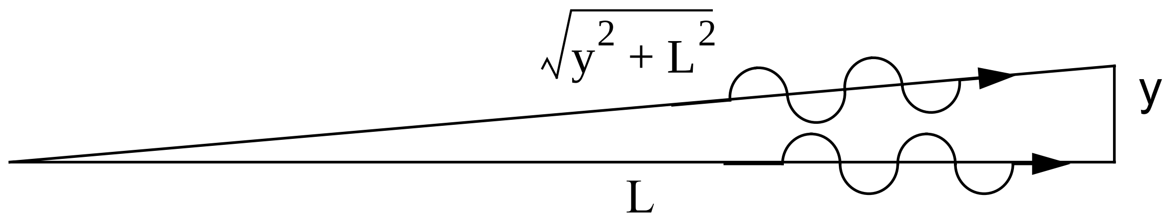
Single-Wavelength Angular Divergence

For an insertion device, the source is extended along the direction of motion by an amount equal to the device length $L = N\,\lambda_u$



For a single wavelength, (e.g. the undulator's first harmonic), destructive interference occurs between two waves separated by an angle θ when the path length difference $\sqrt{y^2 + L^2} - L$ approaches half a wavelength

Diffraction Limited Source Size and Angular Divergence



Path length difference

$$\sqrt{y^2 + L^2} - L = L\left(\sqrt{1 + \frac{y^2}{L^2}} - 1\right) \approx \frac{y^2}{2L} = \frac{\lambda}{2} \Rightarrow y_{\lambda/2} = \sqrt{\lambda L}$$

Even for a zero-emittance electron beam, the photon beam will be observed to have an apparent beam size of approximately $y_{\lambda/2} = \sqrt{\lambda L}$

In addition it will be observed to have an effective angular divergence given by

$$\theta_{\lambda/2} = \frac{y}{L} = \sqrt{\frac{\lambda}{L}}$$

Photon Emittance

A more careful calculation gives the diffraction-limited rms source size and angular divergence

$$\sigma_{ph} = \frac{\sqrt{\lambda L}}{4\pi} \qquad \sigma_{ph'} = \sqrt{\frac{\lambda}{L}}$$

The product of the two yields the diffraction-limited photon emittance

$$\boxed{\varepsilon_{ph} = \sigma_{ph}\sigma_{ph'} = \frac{\lambda}{4\pi}}$$

Undulator Diffraction-Limited Source Size and Angular Divergence

For an undulator, the fundamental wavelength λ_1 is

$$\lambda_1 = \frac{\lambda_{ID}}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

$$\gamma = \frac{E}{mc^2} \qquad K = 0.934\, B_{max}(T)\, \lambda_{ID}(cm) \qquad L = N\lambda_{ID}$$

So

$$\sigma_{ph} = \frac{\sqrt{\lambda_1 L}}{4\pi} = \frac{\lambda_{ID}}{4\pi\gamma}\sqrt{\frac{(1 + K^2/2)N}{2}} \qquad \sigma_{ph'} = \sqrt{\frac{\lambda}{L}} = \frac{1}{\gamma}\sqrt{\frac{1 + K^2/2}{2N}}$$

Overall Photon Beam Size, Convolving Electron Beam Parameters and Diffraction Limited Parameters

$$\sigma_{ph} = \frac{\sqrt{\lambda_1 L}}{4\pi} = \frac{\lambda_{ID}}{4\pi\gamma}\sqrt{\frac{(1+K^2/2)N}{2}}$$

$$\sigma_{ph'} = \sqrt{\frac{\lambda}{L}} = \frac{1}{\gamma}\sqrt{\frac{1+K^2/2}{2N}}$$

$$\sigma_y = \sqrt{\beta_y \varepsilon_y} \qquad \sigma_x = \sqrt{\beta_x \varepsilon_x + \eta^2 \sigma_\delta^2}$$

$$\sigma_{y'} = \sqrt{\gamma_y \varepsilon_y} \qquad \sigma_{x'} = \sqrt{\gamma_x \varepsilon_x + \eta'^2 \sigma_\delta^2}$$

$$\Sigma_y = \sqrt{\sigma_y^2 + (\sigma_{ph})^2} \qquad \Sigma_x = \sqrt{\sigma_x^2 + (\sigma_{ph})^2}$$

$$\Sigma_{y'} = \sqrt{\sigma_{y'}^2 + (\sigma_{ph}')^2} \qquad \Sigma_{x'} = \sqrt{\sigma_{x'}^2 + (\sigma_{ph}')^2}$$

An approximate brightness formula [4] is then given by

$$\boxed{B = \frac{\mathcal{F}_n}{(2\pi)^2 \varepsilon_{xt} \varepsilon_{yt}}}$$

$$\mathcal{F}_n = f_1 \cdot 1.43 \cdot 10^{14}\, N Q_n I$$

I is the circulating current in Amps and $Q_n (< 1)$ is a factor depending on K [4]

B is in units of $photons/(s \cdot mm^2 \cdot mrad^2 \cdot 0.1\% BW)$

On resonance ($\frac{\Delta\omega}{\omega} = 0$): $f_1 = 0.5$ (central cone, highest brightness on axis)

Off resonance ($\frac{\Delta\omega}{\omega} = -\frac{1}{nN}$: $f_1 = 1$ (ring, highest average brightness)

Here $\varepsilon_{xt} = \Sigma_{x'}\Sigma_x$ and $\varepsilon_{yt} = \Sigma_{y'}\Sigma_y$

w. Joho, SLS, http://slsbd.psi.ch/pub/slsnotes/sls0495.ps

[4] K. Kim, AIP 184, 1989, p. 565