

CONSIDERATIONS IN THE DESIGN OF UNDULATORS

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A. Relevant Radiation Formulas

(1) The midplane field in an undulator is given approximately by

$$B_y(s) = B_o \sin \left( 2\pi \frac{s}{\ell} \right)$$

where

y = coordinate in direction of gap

s = coordinate in direction of beam

$\ell$  = undulator period length

$B_o$  = peak field

The orbit wiggles in the transverse direction x perpendicular to y and s, and is given by

$$x' = \frac{\ell}{2\pi\rho_o} \cos \left( 2\pi \frac{s}{\ell} \right) \equiv x'_o \cos \left( 2\pi \frac{s}{\ell} \right)$$

where

$$\text{prime} \equiv \frac{d}{ds}$$

$$\rho_o = \frac{1}{B_o} \text{ (rigidity, } B\rho = \frac{c}{e} p, \text{ of beam).}$$

The deflection parameter K is defined by

$$\begin{aligned} K &\equiv \frac{\text{max. wobble angle}}{\text{radiation angle}} = \frac{x'_o}{1/\gamma} = \frac{\gamma\ell}{2\pi\rho_o} \\ &\approx \frac{e}{2\pi mc^2} B_o \ell = (0.9337 \text{ T}^{-1} \text{ cm}^{-1}) B_o \ell \end{aligned} \quad (1)$$

where  $\gamma$  is the particle energy in rest energy,  $mc^2$ , units. A device with  $K \gtrsim 10$  for which the radiation spectrum is more-or-less continuous is called a wiggler and that with  $K \lesssim 4$  for which the spectrum is discrete is called an undulator. We shall refer to all of them as undulators.

(2) The wavelength  $\lambda_k$  of the forward  $k^{\text{th}}$  harmonic radiation of the undulator is given by

$$\lambda_k = \frac{1}{k} \frac{\ell}{2\gamma^2} \left(1 + \frac{K^2}{2}\right) \quad k = \text{odd} \quad (2)$$

The corresponding energy of the radiation is

$$\begin{aligned} E_k &= \frac{2\pi\hbar c}{\lambda_k} = \frac{1.24 \times 10^{-7} \text{ keV cm}}{\lambda_k} \\ &= (2.48 \times 10^{-7} \text{ keV cm}) \frac{k\gamma^2}{\ell(1 + \frac{K^2}{2})} \end{aligned} \quad (3)$$

(3) The spectral brilliance of the radiation, B, is given by

$$B \equiv \frac{\text{spectral flux (flux/BW)}}{\text{source phase volume } \phi} = \frac{F}{4\pi \Sigma_x \Sigma'_x \Sigma_y \Sigma'_y} \quad (4)$$

where BW means bandwidth. The spectral flux is given by

$$F = \frac{\pi\alpha}{e} NIQ_k = [1.431 \times 10^{14} \text{ sec}^{-1} \text{ mBW}^{-1} \text{ A}^{-1}] NIQ_k \quad (5)$$

where

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137} = \text{fine structure constant}$$

N = no. of periods in undulator

I = particle (electron or positron) beam current

$$Q_k(K) = 4u \left[ \frac{J_{\frac{k-1}{2}}(u)}{2} - \frac{J_{\frac{k+1}{2}}(u)}{2} \right]^2 \quad (6)$$

$$\text{with } u \equiv \frac{kK^2}{4 + 2K^2}$$

mBW  $\equiv$  milli-bandwidth ( $\frac{\Delta\omega}{\omega} = 0.1\%$ )

For the source phase area in either phase plane (x or y) we can write

$$\Sigma^2 \Sigma'^2 = \left( \sigma^2 + \sigma'^2 \frac{L^2}{4} + \frac{\lambda L}{4} \right) \left( \sigma'^2 + \frac{\lambda}{L} \right) \quad (7)$$

$\uparrow$                        $\uparrow$                        $\uparrow$   
 focal depth      diffraction  
 term                      terms

where

$$\sigma = \sqrt{\varepsilon\beta} = \text{rms particle beam width at midpoint of undulator}$$

$$\sigma' = \sqrt{\varepsilon/\beta} = \text{rms particle beam divergence at midpoint of undulator}$$

$$\left( \begin{array}{l} \varepsilon = \text{rms emittance of beam} \\ \beta = \text{minimum amplitude function at midpoint of undulator} \end{array} \right)$$

$$L = N\ell = \text{total length of undulator}$$

$$\lambda = \text{wavelength of radiation}$$

### B. Approximate Magnetics for a Hybrid REC Undulator

If the width (along x) of the undulator is sufficiently large, the field distribution is approximately two-dimensional as shown in Fig. 1. The total magnetic flux emanating from each pole piece is roughly equal to that generated by one block of REC, but only a fraction of it crosses the midplane. This fraction looks as though it originates approximately from a magnetic pole q located at a distance  $a \equiv (\text{half-gap}) + (\text{half-pole thickness}) = (g+t)/2$  from the midplane. The field on the midplane produced by two rows of alternating poles with strengths  $\pm q$  and at distance  $\pm a$  from the midplane are easily derived to be

$$\begin{aligned} B_y(s) &= \frac{16\pi q}{\ell} \frac{\sinh 2\pi \frac{a}{\ell} \sin 2\pi \frac{s}{\ell}}{\cosh 4\pi \frac{a}{\ell} + \cos 4\pi \frac{s}{\ell}} \\ &= \frac{8\pi q}{\ell} \frac{1}{\sinh 2\pi \frac{a}{\ell}} \frac{\sin 2\pi \frac{s}{\ell}}{\cos^2 2\pi \frac{s}{\ell} + \frac{1}{\sinh^2 2\pi \frac{a}{\ell}}} \\ &\approx B_0 \sin 2\pi \frac{s}{\ell} \end{aligned}$$

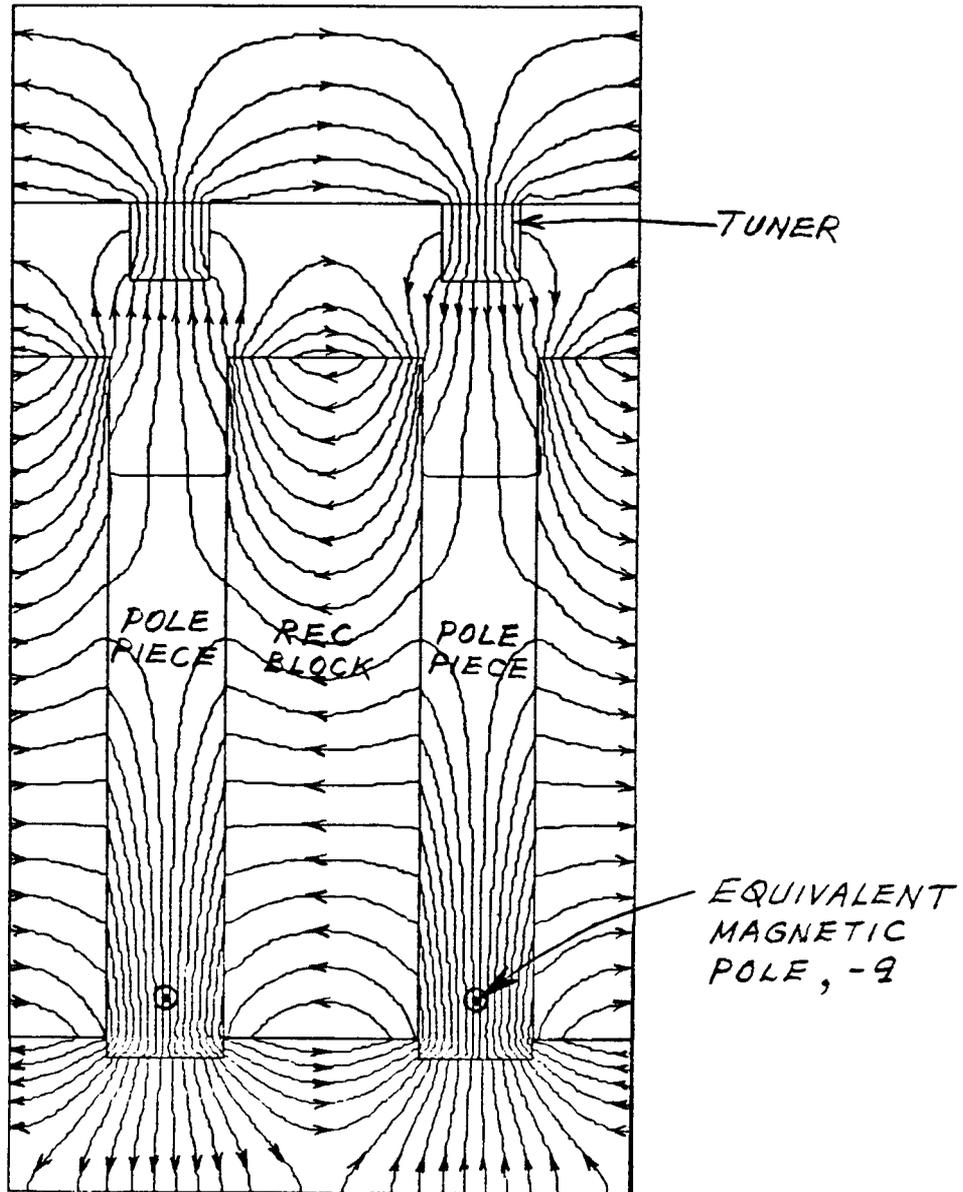


Figure 1. Field distribution in one cell of a hybrid REC undulator.

The last approximate expression results from the fact that the amplitude of the  $\cos^2 2\pi s/\ell$  term in the denominator is generally much smaller than 1 and can be neglected. This then gives

$$B_o \ell = \frac{8\pi q}{\sinh 2\pi \frac{a}{\ell}} = \frac{8\pi q}{\sinh \pi \frac{g+t}{\ell}} . \quad (8)$$

The value of  $q$  depends on the material and mass of the REC magnet block. As an average conventional value, we take

$$8\pi q \sim 3 \text{ T cm}$$

Together with Eqs. (1) and (8), this gives

$$K = \frac{2.8}{\sinh \pi \frac{g+t}{\ell}} \quad (9)$$

### C. Optimization Considerations and Procedures

#### (1) Radiation wavelength $\lambda$

In the following we shall concentrate only on the first harmonic radiation and shall drop the  $k=1$  subscript on all parameters. The first harmonic wavelength is given by Eqs. (2) and (9) to be

$$\lambda = \frac{\ell}{2\gamma^2} \left[ 1 + \frac{3.92}{\sinh^2 \pi \frac{g+t}{\ell}} \right] . \quad (10)$$

The pole piece thickness  $t$  has to be large enough so that the magnetic flux density in the pole piece does not exceed saturation. Generally, with available REC magnets, this is not a problem. Design convenience gives  $t \cong \ell/6$ ; namely, that the REC magnet is about twice as thick as the pole piece. We shall assume this value for all following calculations. The gap  $g$  is generally adjustable from a minimum value upward. With a given beam energy (or  $\gamma$ ) and a desired radiation wavelength  $\lambda$  at a prescribed minimum gap, Eq. (10) gives the necessary period length  $\ell$ . When the magnet gap is increased hence, the peak magnetic field  $B_o$  decreased, the deflection parameter  $K$  decreases and so does  $\lambda$ . This is the simplest and most straightforward way to tune for different wavelengths (or energies) of the radiation from the undulator.

As an example, we consider a high energy and high brilliance undulator on a high energy storage ring. A realistic minimum gap is  $g = 1$  cm. For particle beams of 6 GeV, 7 GeV and 8 GeV to get 20 keV first harmonic radiation ( $\lambda = 6.2 \times 10^{-9}$  cm) the required  $\ell$  values as given by Eq. (10) are listed in Table 1.

**Table 1**  
Parameters of 20 keV First-Harmonic-Energy Tunable Undulators

Beam energy (GeV)	Period lengths $\ell$ (cm) for $E = 20$ keV at $g_{\min} = 1$ cm	Radiation energy $E$ (keV) at $g = \infty$
6.0	1.557	21.97
7.0	1.918	24.27
8.0	2.249	27.03

Also given in the last column are the radiation energies when the gap is opened to infinity, hence  $K$  is reduced to zero. These then give the ranges of radiation energy that can theoretically be reached by tuning the magnet gap  $g$ . One notes, however, that at  $K = 0$ , the radiation flux is zero. It is clear that  $g = \infty$  is unrealistic. Nevertheless, Table 1 shows that the tuning range is greater for higher beam energy.

(2) Radiation spectral flux

The spectral flux for the first harmonic radiation is given by Eqs. (5) and (6) to be proportional to

$$\begin{aligned}
 Q(K) &= 4u[J_0(u) - J_1(u)]^2 \\
 &\cong 4u\left(1 - \frac{u}{2} - \frac{u^2}{4}\right)^2
 \end{aligned} \tag{11}$$

where

$$u = \frac{K^2}{4 + 2K^2}$$

and where the second expression of  $Q(K)$  is a good approximation for small values of  $u$ , hence for all values of  $K$ , since  $u$  goes up to only 0.5 when  $K$  goes to infinity. Both the exact and the approximate expressions are plotted in Fig. 2 which shows the excellence of the approximation, especially for values of  $K < 1$ .

The graph also shows that  $Q$  is a rather steep function of  $K$  for  $K < 1$ . When one aims for the highest possible radiation energy (smallest  $\lambda$ ) with a given beam energy (or  $\gamma$ ) by reducing  $g$  and  $\ell$  to the minimum, the range of radiation energy tunable by varying  $g$  becomes very small. With a small initial value of  $K$ , which falls on the very steep part of the  $Q$  curve, the small reduction of  $K$  obtained by increasing  $g$  (reducing  $B_0$ ) raises  $E$  only slightly, but reduces  $Q$  and hence the spectral brilliance rather precipitously.

In practice, the principal parameters determining the design of a high energy tunable undulator are the minimum gap  $g_0$  and the maximum desired radiation energy  $E_1$ . At the minimum gap, the radiation energy  $E_0$  is given by

$$E_0 = (2.48 \times 10^{-7} \text{ keV cm}) \frac{\gamma^2}{\ell} (1 - 2u_0) \quad (12)$$

and the spectral brilliance is proportional to

$$Q_0 = 4u_0 \left(1 - \frac{u_0}{2} - \frac{u_0^2}{4}\right)^2 \quad (13)$$

where  $u_0$  is expressed in terms of  $g$  (and  $\ell$ ) by

$$\begin{aligned} 2u_0 &= 1 - \left(1 + \frac{K_0^2}{2}\right)^{-1} \\ &= \left[1 + \frac{3.92}{\sinh^2 \pi \frac{g_0 + t}{\ell}}\right]^{-1} \end{aligned} \quad (14)$$

46 1320

K<sub>2</sub>E 10 X 10 TO 1/2 INCH 7 X 10 INCHES  
KEUFFEL & ESSER CO. MADE IN U.S.A

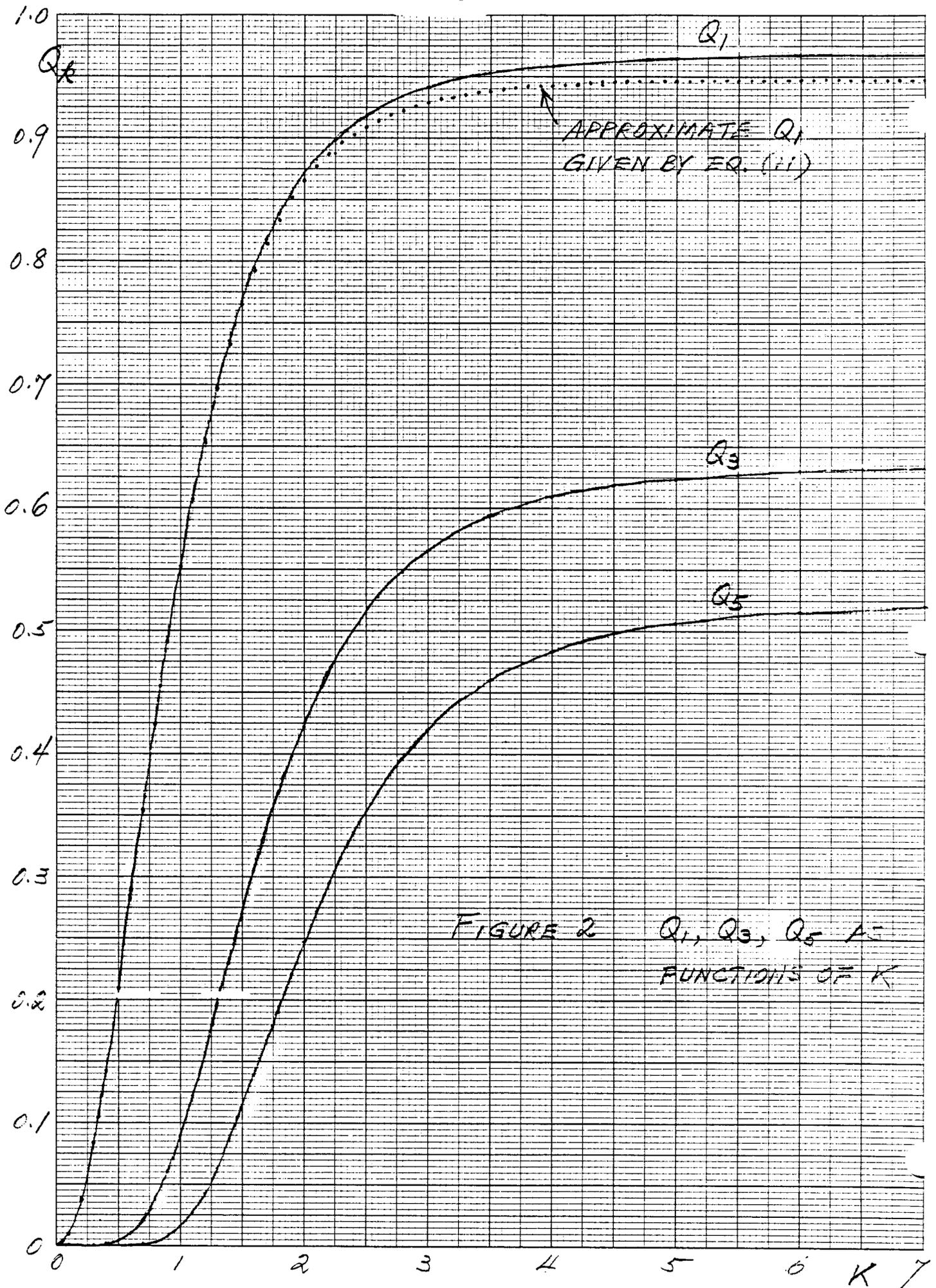


FIGURE 2  $Q_1, Q_3, Q_5$  AS  
FUNCTIONS OF  $K$

The maximum radiation energy  $E_1$  is obtained at the widest gap  $g_1$  when the reduced brilliance is at the minimum allowable value. The minimum brilliance is proportional to

$$Q_1 = 4u_1 \left(1 - \frac{u_1}{2} - \frac{u_1^2}{4}\right)^2 \quad (15)$$

where now  $u_1$  is expressed in terms of  $E_1$  (and  $\ell$ ) by

$$\begin{aligned} 2u_1 &= 1 - \left(1 + \frac{K_1^2}{2}\right)^{-1} \\ &= 1 - \frac{\ell E_1}{(2.48 \times 10^{-7} \text{ keV cm})\gamma^2} \end{aligned} \quad (16)$$

It remains now to determine the allowed reduction in  $Q$  at the top energy. If  $Q_1$  must, at the minimum, be a fraction  $f$  of  $Q_0$ , the equation

$$Q_1 = fQ_0 \quad (17)$$

containing only  $\ell$  as variable will yield the period length required. In Table 2 we give the parameters so derived of two adjoining high energy undulators assuming an allowable brilliance reduction factor  $f = 1/2$  and for the three beam energies studied before. One sees that the radiation energy range that can be covered by the two adjoining undulators increases from 2.40 keV at 6 GeV to 5.64 keV at 7 GeV to 9.45 keV at 8 GeV showing that for good tuning range at the maximum desired radiation energy, one should use somewhat higher-than-minimum necessary beam energy so that the period length  $\ell$  does not have to be pushed to the absolute minimum.

**Table 2**

Parameters of sets of two adjoining tunable undulators with 20 keV maximum radiation energy for three different beam energies

Beam energy	Undulator I		Undulator II	
	Open gap	Min. gap	Open gap	Min. gap
<u>6 GeV</u>				
$l$ (cm)		1.623		1.704
$g$ (cm)	1.199	1.000	1.214	1.000
$E$ (keV)	20.00	18.87	18.87	17.60
$Q$	0.0987	0.1975	0.1154	0.2307
<u>7 GeV</u>				
$l$ (cm)		2.088		2.332
$g$ (cm)	1.292	1.000	1.350	1.000
$E$ (keV)	20.00	17.40	17.40	14.36
$Q$	0.1948	0.3896	0.2401	0.4801
<u>8 GeV</u>				
$l$ (cm)		2.579		3.154
$g$ (cm)	1.417	1.000	1.593	1.000
$E$ (keV)	20.00	15.58	15.58	10.55
$Q$	0.2795	0.5589	0.3459	0.6918

(All  $g/l$  values are within the "Halbach limit" of 0.7.)

### (3) Source phase volume

To maximize the spectral brilliance, we want to minimize the source phase volume  $\Phi$  given in Eqs. (4) and (7). This can be written as

$$\begin{aligned}
 \Phi &= 4\pi^2 \Sigma_x \Sigma_x' \Sigma_y \Sigma_y' \\
 &= 4\pi^2 \left( \sigma_x^2 + \sigma_x'^2 \frac{L^2}{4} + \frac{\lambda L}{4} \right)^{1/2} \left( \sigma_x'^2 + \frac{\lambda}{L} \right)^{1/2} \\
 &\quad \times \left( \sigma_y^2 + \sigma_y'^2 \frac{L^2}{4} + \frac{\lambda L}{4} \right)^{1/2} \left( \sigma_y'^2 + \frac{\lambda}{L} \right)^{1/2} \\
 &= 4\pi^2 \epsilon_x \epsilon_y G_x G_y
 \end{aligned} \tag{18}$$

with  $G_x$  and  $G_y$  defined by

$$G \equiv \left[ 1 + \frac{\lambda}{\epsilon} \frac{\beta}{L} + \frac{1}{4} \left( \frac{L}{\beta} + \frac{\lambda}{\epsilon} \right)^2 \right]^{1/2} \quad (\text{for either } x \text{ or } y) \quad (19)$$

The term  $\beta/L$  exhibits the effect of the focal depth. To minimize the beam spot (or  $G$ ) over the whole length of the undulator  $\beta$  should be approximately equal to  $L$ . This is evident from the fact that  $\beta/L$  appears in the expression for  $G$  both right-side-up and in the inverse, and can further be substantiated by direct calculation. Thus, we set  $\beta = L$  and obtain

$$\begin{aligned} G_{\min} &= \left( \frac{5}{4} + \frac{6}{4} \frac{\lambda}{\epsilon} + \frac{1}{4} \frac{\lambda^2}{\epsilon^2} \right)^{1/2} \\ &= \frac{1}{2} \left[ \left( 5 + \frac{\lambda}{\epsilon} \right) \left( 1 + \frac{\lambda}{\epsilon} \right) \right]^{1/2} \end{aligned} \quad (20)$$

The parameter  $\frac{\lambda}{\epsilon}$  gives the contribution from diffraction. This expression for  $G_{\min}$  shows that for a given wavelength  $\lambda$  the beam emittance  $\epsilon$  should not be much smaller than  $\lambda$ , otherwise the source size will be excessively enlarged by diffraction. In practice, for nearly all cases of interest  $\lambda/\epsilon < 1$ . Thus,  $G_{\min}$  lies somewhere in the narrow range of  $\sqrt{5}/2 = 1.1$  (at  $\lambda/\epsilon = 0$ ) to  $\sqrt{3} = 1.7$  (at  $\lambda/\epsilon = 1$ ). If we set approximately

$$G_x \cong G_y \cong \frac{3}{2}$$

and

$$\begin{cases} \epsilon_x = \frac{1}{1+k} \epsilon_o \\ \epsilon_y = \frac{k}{1+k} \epsilon_o \end{cases} \quad (21)$$

where  $\epsilon_o$  is the natural emittance and  $k$  is the horizontal-vertical coupling fraction of the beam, Eq. (18) gives

$$\Phi = 9\pi^2 \frac{k}{(1+k)^2} \epsilon_o^2 \quad (22)$$

which together with Eqs. (4) and (5) gives, finally, for the spectral brilliance

$$B = \frac{(1.431 \times 10^{14})NIQ}{9\pi^2 \frac{k}{(1+k)^2} \epsilon_0^2} \quad (23)$$

As an example, we take the 20 keV undulator designed for the Argonne 6 GeV storage ring with the following parameters:

Beam energy	= 6 GeV
$\epsilon_0$	= $7.8 \times 10^{-9}$ m-rad = $7.8 \times 10^{-3}$ mm-mrad
k	= 0.1
I	= 100 mA = 0.1 A
L	= 5 m (N = 500 cm/ $\ell$ )

and with  $\ell$  and Q as given in Table 2. The undulator design and performance parameters derived from the above considerations and formulas are:

### Undulator I

$$\ell = 1.623 \text{ cm}$$

$$N = 308$$

$$B = 9.868 \times 10^{18} \text{ Q}$$

	<u>Open gap</u>	<u>Minimum gap</u>
g(cm)	1.199	1.000
E(keV)	20.00	18.87
Q	0.0987	0.1975
B(photon/sec/mBW/mm <sup>2</sup> /mrad <sup>2</sup> )	$0.974 \times 10^{18}$	$1.949 \times 10^{18}$

Undulator II

$$\ell = 1.704 \text{ cm}$$

$$N = 293$$

$$B = 9.388 \times 10^{18} \text{ Q}$$

	<u>Open gap</u>	<u>Minimum gap</u>
g(cm)	1.214	1.000
E(keV)	18.87	17.60
Q	0.1154	0.2308
B(photon/sec/mBW/mm <sup>2</sup> /mrad <sup>2</sup> )	$1.083 \times 10^{18}$	$2.166 \times 10^{18}$

We see that even at the open gap settings, the halved brilliance is still about  $1 \times 10^{18}$  photon/sec/mBW/mm<sup>2</sup>/mrad<sup>2</sup>.