THEORETICAL ESTIMATION OF THE DYNAMIC APERTURE
FOR A CHASMAN–GREEN LATTICE

If \( v_x \) is close enough to \( v_x = 1 \) per period, the Hamiltonian for the particle motion in a lattice with reflective symmetry can be written as

\[
H = \sqrt{8} J_x \left[ J_{x,33} \cos Q_{33} + (3 J_{x,11} - 6 J_y B_{11}) \cos Q_{11} \right]
\]

where \( Q_{jm} = j (\phi_x - \psi_x) + (jv_x - m)\theta \)

\( \phi_x \) is the particle phase,
\( \psi_x \) is the lattice phase,
and \( \theta \) is the angular coordinate along the orbit.

Using a reflective symmetry point as reference, the harmonic components are given by

\[
A_{jm} = \sum_k \frac{S_k}{48\pi} \cos \left( j\psi_x - (jv_x - m)\theta \right)
\]

\[
B_{11} = \sum_k \frac{S_k}{48\pi} \cos \left( \psi_x - (v_x - 1)\theta \right)
\]

where

\[
S_k = \left( \frac{B_{11}}{B_{31/2}} \right)_k, \quad \bar{S}_k = \left( \frac{B_{11}}{B_{11/2}} \right)_k
\]

and \( \frac{B''(x)\ell}{B_p} \) are the sextupole strengths.
Since
\[ H'(\theta) = -\sqrt{8} \left( v_x - 1 \right) J_{\chi}^{1/2} \left[ 3 A_{33} J_{\chi} \sin Q_{33} + 3 J_{x} A_{11} - 6 J_{y} B_{11} \right] \sin Q_{11} \]
and
\[ J'(\theta) = \sqrt{8} J_{\chi}^{1/2} \left[ 3 A_{33} J_{\chi} \sin Q_{33} + 3 J_{x} A_{11} - 6 J_{y} B_{11} \right] \sin Q_{11} \]
the function
\[ (v_x - 1) J_{\chi} + H = C \text{ (a constant)} \]
\[ (J_y \text{ is also a constant}) \]

We define
\[ \Delta = v_x - 1 \]
\[ N_x = \sqrt{\frac{2 J_{\chi}}{\epsilon}} \]
\[ N_y = \sqrt{\frac{4 J_y}{\epsilon}} \text{, (constant)} \]

where \( \epsilon \) is the natural emittance.

In these variables
\[ \frac{\Delta \epsilon N_x^2}{2} + \epsilon^{3/2} N_x \left[ N_x^2 A_{33} \cos Q_{33} + 3(N_x^2 A_{11} - N_y^2 B_{11}) \cos Q_{11} \right] = C \]

Define further
\[ x_o = \frac{A}{2 \sqrt{\epsilon A}} \]
\[ D = \frac{3 B_{11}}{A} \]
and
\[ f(N_x) = N_x^3 + N_x^2 x_o - N_x N_y^2 D \]

where \( A = 3 A_{11} + A_{33} \).
The constant of the motion is obtained from the initial conditions $N_{x0}$ and $\Phi_x = \Psi_x = \Theta = 0$. As $Q_{33}$, $Q_{11}$ vary with the particle motion in the lattice $N_x$ is determined by the constant $C$. When $\cos Q_{33} = \cos Q_{11} = -1$, the particle has its maximum displacement in the opposite direction from the starting displacement. This displacement is obtained by finding the negative solution of the equation

$$f(N_x) = f(N_{x0}).$$

The functional form of $f(N_x)$ for $X_o > 0$ is shown in the sketch.

One has

$$N_{x1} = -\frac{X_o}{3} (1+B)$$

$$N_{x2} = -\frac{X_o}{3} (1-B)$$

$$N_{x3} = -\frac{X_o}{3} (1-2B)$$

$$B = \sqrt{1 + \frac{3N_x}{X_o^2} \frac{x^2}{y^2}}$$
For an initial condition \( N_{x_0} \), the displacement of the particle ranges between \( a \) and \( b \) as shown on the sketch. For \( N_{x_0} > N_{x_3} \), \( b \) does not exist and the motion is unstable. The limit of stability occurs for \( N_{x_0} = N_{x_3} \), and the displacement ranges from \( N_{x_3} \) to \( N_{x_1} \).

For \( D < 0 \), the stable region \( (N_{x_3} - N_{x_1}) \) decreases as \( N_y^2 \) increases.

For

\[
N_y^2 = N_{y_{\text{max}}}^2 = \frac{X_o^2}{3D},
\]

the stable region becomes 0. For \( N_y > N_{y_{\text{max}}} \), there is no stability.

(For \( D > 0 \), the stable region increases as \( N_y \) increases.)

Figure 1 shows the comparison between the predicted stability region and the results of tracking for the 7-GeV Advanced Photon Source CDR lattice.\(^{(1)}\) The tracking results are obtained by searching for the limit of stability in \( N_y \) for a fixed value for \( N_x \). (Negative \( N_x \) is understood to mean \( \cos \phi_{x_0} = -1 \).)

The limits of the stable region for this lattice are caused by the chromaticity correcting sextupoles and the predominance of the integral resonance for \( v_x = 0.88 \) per period. The values used in predicting the results are

\[
\begin{align*}
\epsilon &= 8.08 \times 10^{-9} \text{ m} \\
A_{33} &= -1.5200 \text{ m}^{-1/2} \\
A_{11} &= -1.3445 \text{ m}^{-1/2} \\
B_{11} &= 1.4385 \text{ m}^{-1/2}
\end{align*}
\]

Reference

Figure 1
Dynamic aperture obtained by tracking (solid curve) and predicted by first-order resonance (dotted curve) with chromaticity-correcting sextupoles only.