Optimization of $\beta$ Functions through Insertion Devices

Introduction

It is generally noted that at an undulator straight section, the horizontal beta function is made to be large while the vertical beta function is relatively small. On the other hand, at a wiggler straight section, both horizontal and vertical beta functions are made to be small. In this note we describe a procedure with which optimum settings of the beta functions in the insertion straight section are to be determined. For this we consider separately for the undulator radiation and the radiation from the wiggler device.

Since the brilliance of radiation is a canonically conserved quantity, we use the brilliance as a figure of merit for the consideration. Then the optimization process is to find a set of horizontal and vertical beta functions which would give the maximum brilliance when the natural emittance, emittance coupling constant, length of insertion device and the photon energy to which the insertion device is optimized are specified. The next step of the study is to find sensitivities of the beta functions to the brilliance.

After having done these, we will attempt to find a set of universally optimized beta functions with which all of given kind of the insertion device, e.g., all undulators or all wigglers can be operated. The purpose of this attempt is that if all undulators, regardless of photon energies for which these are optimized, can have the same beta functions, the machine lattice becomes quite simpler and can maintain higher periodicity.

In order to make the description of this study complete, we review some of definitions used herein.

Coupling Constant $k$: Coupling constant is defined to be

$$\varepsilon_x = \varepsilon_{xo}/\sqrt{1+k}$$
$$\varepsilon_y = k\varepsilon_{xo}/\sqrt{1+k}$$
where $\epsilon_{x_0}$ is the natural emittance of the storage ring, and $\epsilon_x$ and $\epsilon_y$ are the resultant horizontal and vertical emittances due to the coupling of $k$, respectively.

Photon flux is defined as number of photons/sec/eV of photon or number of photons/sec/0.1% BW.

Brightness or spectral brightness is defined as

$$\text{Brightness} = \frac{\text{Photon flux / apparent source rms solid angle}}{\text{number of photons/sec}}$$

$$= \frac{\text{number of photons/sec}}{\Omega \ (0.1\% \ BW)}$$

where $\Omega$ is the apparent rms solid angle of the source:

$$\Omega = 2\pi S_x S_y \ \text{(mrad**2)}$$

and

$$S_{x,y} = \sqrt{\sigma_{x,y}^2 + \lambda/L}$$

Here, $S_x$ and $S_y$ are the apparent divergences of the radiation from the insertion device in the $x$ and $y$ directions, and $\sigma_x$ and $\sigma_y$ are the rms divergences of the electron beam in the $x$ and $y$ planes through the insertion device, and $\lambda$ is the wavelength of the photon beam.

$$\sigma_x = \frac{\sqrt{\epsilon_x / \beta_x \ x}}{}$$

$$\sigma_y = \frac{\sqrt{\epsilon_y / \beta_y \ y}}{}$$

The brilliance is then defined to be:

$$\text{Brilliance} = \frac{\text{number of photons/sec}}{S \times \Omega \ (0.1\% \ BW)}$$

where $S$ is the rms source area,
\[ S = 2\pi S_x S_y \]
\[ S_x = \sqrt{\sigma_x^2 + (\lambda L + \sigma_x^2 L^2)/4} \]
\[ S_y = \sqrt{\sigma_y^2 + (\lambda L + \sigma_y^2 L^2)/4} \]
\[ \sigma_x = \sqrt{\beta \varepsilon_x} \]
\[ \sigma_y = \sqrt{\beta \varepsilon_y} \]

**Undulator Optimization:**

Under the given parameters of \( \varepsilon_{x_0} \), \( k \), and \( \lambda \), the goal is to obtain the minimum value of \( S \cdot \Omega \) as a function of beta-x and beta-y. In order to demonstrate the sensitivities \( S \cdot \Omega \) which is the four dimensional transverse phase space with respect to the beta functions, we make contour plots of \( 1/(S \cdot \Omega) \) as a function of beta-x and beta-y.

Figures 1 - 6 show the inverse of the phase spaces for various photon energies (1 ~ 20 keV). The contours are normalized to the maximum value which is marked with an "H", and each contour line is plotted with an interval of 2% from the previous. In another word, the inner most contour represents 98 percentile and the outer contour represents the 90 percentile contour. Also shown in the figures is the phase space value at the peak in unit of meter squared. The values at null beta functions are calculated at beta = 0.1 m.

Figure 7 shows the result of summing all six contours and renormalized by dividing by 6. This is an attempt to find a universal setting of the beta functions with which all undulator could be efficient.

**Wiggler Consideration**

Detailed consideration of the brightness of wiggler shows that

\[ \text{Brightness} = 2N \cdot 3.461 \times 10^6 \gamma^2 I \left( \frac{\varepsilon}{\varepsilon_c} \right)^2 k_{2/3}^2 \left( \frac{\varepsilon}{2\varepsilon_c} \right) \]

which is independent of the wiggler geometry. Therefore, to optimize the
brilliance of wiggler, we optimize the quantity $S$. We have studied this for the photon energies up to 40 keV, and the results are shown in Figure 8. Notice that the brilliance for the wiggler is independent of the photon energy.

**Conclusion**

This simple study shows that for about 5 meter undulator, operating with $\epsilon x_0 = \gamma \lambda 10^{-9}$ m and $k = 0.1$, $\beta_x = 12$ m and $\beta_y = 6$ m would provide an optimum undulator radiations from 1 - 20 keV range. For wiggler radiation, the beta function setting should be around 2 m for all photon energies.
UNDULATOR PHASE SPACE CALCULATION

Epsx, Epsy, K2  7.27272727272E-9  7.27272727272E-10  .1
Photon Wave Length: in Å and in keV  12.397  1
Insertion Device Length  5.2
Minimum Value of Phs at Bx By  1.34603E-17  8.  5.

Horizontal Axis: Betax; Vertical Axis: Betay
Plot is normalized with Phase Space minimum

Figure 1
UNDULATOR PHASE SPACE CALCULATION
Epsx, Epsy, K2  7.27272727273E-9  7.27272727273E-10 .1
Photon Wave Length: in A and in keV  2.4794  5
Insertion Device Length  5.2
Minimum Value of Phs at Bx By  7.45302E-18  13.  7.

Horizontal Axis Betax: Vertical Axis Betay
Plot is normalized with Phase space minimum

Figure 2
UNDULATOR PHASE SPACE CALCULATION

Epsx, Epsy, K2 7.2727272727E-9 7.2727272727E-10 1
Photon Wave Length: in Å and in keV 1.2397 10
Insertion Device Length 5.2
Minimum Value of Phs at Bx By 6.57951E-18 16.8

Horizontal Axis Betax: Vertical Axis Betay
Plot is normalized with Phase Space minimum

Horizontal Axis (right): Betay between 0. 20.
Vertical Axis (down) : Betay between 0. 20.

Figure 3
UNDULATOR PHASE SPACE CALCULATION

\[ \text{Epsx, Epsy, K2} \quad 7.27272727273E-9 \quad 7.27272727273E-10 \quad .1 \]

Photon Wave Length: in Å and in keV \[ 0.825455555557 \quad 15 \]

Insertion Device Length \[ 5.2 \]

Minimum Value of Phs at Bx By \[ 6.25298E-18 \quad 19.9 \]

Horizontal Axis Betax: Vertical Axis Betay
Plot is normalized with Phase Space minimum

---

**Figure 4**
UNDULATOR PHASE SPACE CALCULATION
Epsx, Epsy, K2 7.27272727273E-9 7.27272727273E-10 .1
Photon Wave Length: in Å and in keV .729235294118 17
Insertion Device Length 5.2
Minimum Value of Phs at Bx By 6.17140E-18 19.

Horizontal Axis (left): Betax; Vertical Axis (right): Betay
Plot is normalized with Phase Space minimum

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Minimum: 50; Maximum: 100; Contour Interval: 2.
Horizontal Axis (right): Betax between 0. 20.
Vertical Axis (down): Betay between 0. 20.

Figure 5
UNDULATOR PHASE SPACE CALCULATION
Epsx, Epsy, K2 7.27272727273E-9 7.27272727273E-10 .1
Photon Wave Length: in A and in keV .61985 .20
Insertion Device Length 5.2
Minimum Value of Phs at Bx By 6.07539E-18 20. 10.

Horizontal Axis Betax: Vertical Axis Betay
Plot is normalized with Phase Space minimum

Figure 6
Figure 7. Summed Contour Plot of Fig. 1-6.
WIGGLER RADIATION PHASE SPACE CALCULATION
Epsx, Epsy, K2 7.27272727273E-9 7.27272727273E-10 .1
Photon Wave Length: in Å and in keV 309925 40
Insertion Device Length 5
Minimum Value of Phs at Bx By 1.17417E-08 2.

Horizontal Axis Bx: Vertical Axis By
Plot is normalized with Phase Space minimum

Figure 8