LS-17
L. Teng
March 18, 1985

MINIMUM EMITTANCE LATTICE FOR SYNCHROTRON RADIATION STORAGE RINGS

L. C. Teng

ANL/FNAL

The natural emittance of an electron beam in a storage ring is given by (see e.g., M. Sands, SLAC 21)

$$\varepsilon = \frac{Cq}{J_x} \gamma^2 \frac{\langle H \rangle}{\rho} \tag{1}$$

where $Cq = = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.832 \times 10^{-13}$ m

J_x = partition factor in the bending plane

γ = total energy in mc^2 units

ρ = orbit radius in bending magnets (assumed the same in all magnets)

$H \equiv \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2$

(α, β, γ = betatron functions
η, η' = dispersion functions)

$\langle\rangle$ = averaging over bending magnets

We shall calculate $\langle H \rangle$ for each bending magnet, then average over all magnets.

A. General Expression for H

This can be calculated in a straightforward manner, but we can save a great deal of arithmetic with some preliminary formal analytical reduction. Defining

$$J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \text{betatron imaginary unit matrix}$$

and

$$Y = \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \text{dispersion vector}$$

we can then write

$$H = \bar{Y}JY = \gamma\eta^2 + \alpha\alpha\eta\eta' + \beta\eta'^2 \tag{2}$$

where

$$\bar{Y} \equiv \tilde{Y}\tilde{S} = \text{symplectic conjugate of } Y$$

$$S \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \text{symplectic unit matrix}$$

~ means transposition

Let s = coordinate along closed orbit, and let $\alpha = 0$ or

$$J = J_o = \begin{pmatrix} 0 & \beta_o \\ -\frac{1}{\beta_o} & 0 \end{pmatrix} \tag{3}$$

and

$$Y = Y_o = \begin{pmatrix} \eta_o \\ \eta'_o \end{pmatrix} \tag{4}$$

at $s = 0$. Taking the bending magnet to be a gradient magnet, we can write the transfer matrix from $s = 0$ to s as

$$M = \begin{pmatrix} \cos\frac{k}{\rho}s & \frac{\rho}{k}\sin\frac{k}{\rho}s \\ -\frac{k}{\rho}\sin\frac{k}{\rho}s & \cos\frac{k}{\rho}s \end{pmatrix} \tag{5}$$

where
$$\begin{cases} \rho = \text{bending radius,} \\ k \equiv \sqrt{1 + \rho\frac{B'}{B}} \quad \text{and} \\ \frac{B'}{B} = \text{relative field gradient} \end{cases} \tag{6}$$

In the following, all lengths will be expressed in units of ρ/k unless otherwise specified. In these units, we have

$$M = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \tag{7}$$

The general betatron and dispersion functions are then given by

$$\begin{cases} J = MJ_oM^{-1} = MJ_o\bar{M} \\ Y = MY_o + X \qquad \text{with} \qquad X \equiv \frac{1}{k}\begin{pmatrix} 1-\cos s \\ \sin s \end{pmatrix} \end{cases}$$

where the symplectic conjugate of M is defined by

$$\bar{M} \equiv S\tilde{M}\tilde{S} = \tilde{S}\tilde{M}S = M^{-1}.$$

With these, we can then write

$$H = \bar{Y}JY = (\bar{Y}_o\bar{M} + \bar{X})MJ_oM^{-1}(MY_o + X)$$

$$= \bar{X}MJ_oM^{-1}X + 2\bar{X}MJ_oY_o + \bar{Y}_oJ_oY_o$$

$$= \frac{1}{k^2}[\beta_o \sin^2 s + \frac{1}{\beta_o}(1 - \cos s)^2]$$

$$+ \frac{2}{k}[\beta_o \eta_o' \sin s - \frac{1}{\beta_o} \eta_o (1 - \cos s)]$$

$$+ [\beta_o \eta_o'^2 + \frac{1}{\beta_o} \eta_o^2] \tag{8}$$

or

$$k^2 H = \beta_o (k\eta_o' + \sin s)^2 + \frac{1}{\beta_o}(1 - k\eta_o - \cos s)^2 \tag{9}$$

B. Averaging and Minimizing

Case A - the most general case.

We assume that the magnet has length ℓ and has its midpoint situated at $s = s_o$ (magnet from $s_o - \frac{\ell}{2}$ to $s_o + \frac{\ell}{2}$). Performing the averaging, we get

$$k^2 \langle H \rangle = \frac{k^2}{\ell} \int_{s_o - \frac{\ell}{2}}^{s_o + \frac{\ell}{2}} H \, ds$$

$$= \beta_o [A + B\cos 2s_o + (k\eta_o' + S_{1/2} \sin s_o)^2]$$

$$+ \frac{1}{\beta_o}[A - B\cos 2s_o + (1 - k\eta_o - S_{1/2}\cos s_o)^2] \tag{10}$$

where

$$\left\{ \begin{array}{l} A = \frac{1}{2} - \frac{1}{\ell} C_1 \ , \quad B = \frac{1}{\ell} C_1 - \frac{1}{2} S_1 \\ \\ \text{with } S_n \equiv \frac{\sin n\ell}{n\ell} \ , \quad C_n \equiv \frac{1 - \cos n\ell}{n\ell} \end{array} \right. \tag{11}$$

To minimize $k^2 \langle H \rangle$ with respect to all the parameters, we see first that the dispersion functions at $s = 0$ must be given by

$$\left\{\begin{aligned} k\eta_o' &= - S_{1/2} \sin s_o \\ 1 - k\eta_o &= S_{1/2} \cos s_o \end{aligned}\right. \tag{12}$$

This gives

$$k^2\langle H\rangle = \left(\frac{1}{\beta o} + \beta o\right)A - \left(\frac{1}{\beta o} - \beta o\right)B \cos 2s_o$$

Since within the ranges of interest of all the parameters the coefficient of $\cos 2s_o$ is smaller in magnitude than the first term on the right-hand side, $k^2\langle H\rangle$ is minimized at

$$s_o = 0 \text{ (magnet centered at } s = 0) \tag{13}$$

This gives

$$k^2\langle H\rangle = \beta_o(A + B) + \frac{1}{\beta_o}(A - B) \tag{14}$$

The overall minimum value of

$$k^2\langle H\rangle_{min} = 2\sqrt{A^2 - B^2} \tag{15}$$

is then obtained at

$$\beta_o = \sqrt{\frac{A-B}{A+B}} \ . \tag{16}$$

With $s_o = 0$, the dispersion functions at $s = 0$ given by Eq. (12) become

$$k\eta_o' = 0, \quad k\eta_o = 1 - S_{1/2} \tag{17}$$

Another quantity of interest is the β value, β_e, at the magnet ends. This is given by

$$\beta_e = \sqrt{\frac{A-B}{A+B}} \cos^2 \frac{\ell}{2} + \sqrt{\frac{A+B}{A-B}} \sin^2 \frac{\ell}{2} = \frac{A-B\cos\ell}{\sqrt{A^2-B^2}} \tag{18}$$

This optimal arrangement is shown in Fig. 1

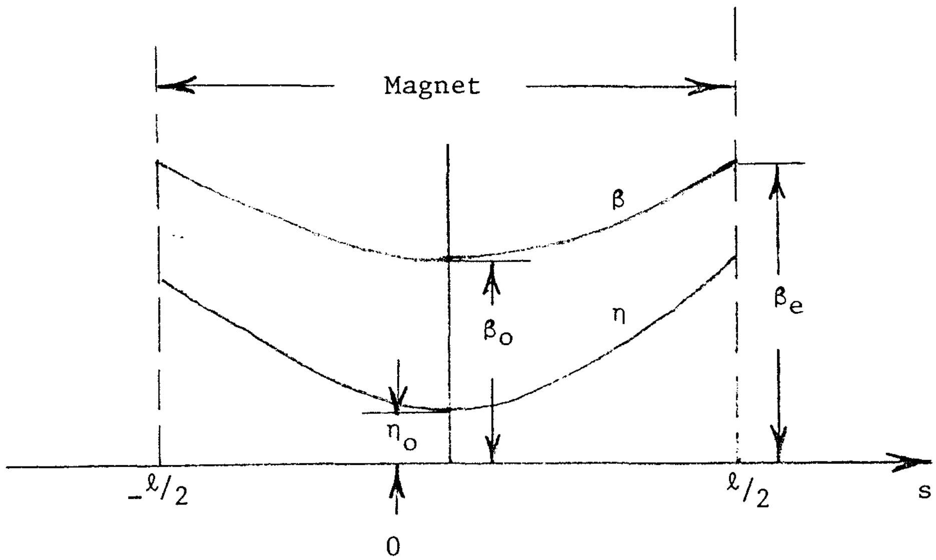

Figure 1. The optimal betatron and dispersion functions which give minimum emittance.

The magnet length ℓ is usually small, and one gets good approximations by expanding to the lowest order terms in ℓ. The minimal $k^2\langle H\rangle$ given by Eq. (15) is

$$k^2\langle H\rangle_{min} = 2\sqrt{A^2 - B^2} \cong \frac{1}{12\sqrt{15}}\,\ell^3 \tag{19}$$

obtained under the conditions

$$\left\{\begin{aligned} & k\eta_o' = 0, \quad k\eta_o = 1 - S_{1/2} \cong \frac{1}{24}\ell^2 \\ & s_o = 0 \\ & \beta_o = \sqrt{\frac{A-B}{A+B}} \cong \frac{1}{2\sqrt{15}}\ell \\ & \beta_e = \frac{A-B\cos\ell}{\sqrt{A^2-B^2}} \cong \frac{8}{\sqrt{15}}\ell \end{aligned}\right. \tag{20}$$

These approximations are very good for $\ell < 1$ as shown by the following comparisons:

	$\ell = 0.05$	$\ell = 0.10$	$\ell = 0.50$
$k^2\langle H\rangle_{min}$	2.7001×10^{-6}	2.1503×10^{-5}	0.0026609
$\frac{1}{12\sqrt{15}}\ell^3$	2.6896×10^{-6}	2.1517×10^{-5}	0.0026896
β_o	0.0064811	0.012908	0.064665
$\frac{1}{2\sqrt{15}}\ell$	0.0064550	0.012910	0.064550
$k\eta_o$	0.00010416	0.00041662	0.010384
$\frac{1}{24}\ell^2$	0.00010417	0.00041667	0.010417

Case A1 - similar to Case A but with $\eta_o = 0$.

In some arrangements it is easier to get $\eta_o = 0$ instead of the value given in Eq. 17. For this we insert $\eta_o' = \eta_o = 0$ in Eq. 10 and minimize $k^2\langle H\rangle$ with respect to s_o and β_o. By the same reasoning, it is easy to see that the minimum value

$$k^2\langle H\rangle_{min} = 2\sqrt{A_1^2 - B_1^2} \cong \frac{1}{8\sqrt{15}}\,\ell^3 \tag{21}$$

is obtained with

$$s_o = 0 \qquad \text{and} \qquad \beta_o = \sqrt{\frac{A_1-B_1}{A_1+B_1}} \cong \frac{3}{4\sqrt{15}}\,\ell \tag{22}$$

where

$$A_1 = 1 - S_{1/2} \qquad \text{and} \qquad B_1 = \frac{1}{2}(2S_{1/2} - S_1 - 1) \tag{23}$$

The end value of β is, again

$$\beta_e = \frac{A_1 - B_1\cos\ell}{\sqrt{A_1^2 - B_1^2}} \cong \frac{23}{4\sqrt{15}}\,\ell \tag{24}$$

In all these formulas we have also put down the approximate expressions obtained by expanding to the lowest order terms in ℓ. The comparison between the exact and the approximate values are given below:

	$\ell = 0.05$	$\ell = 0.10$	$\ell = 0.50$
$k^2\langle H\rangle_{min}$	4.0412×10^{-6}	3.2259×10^{-5}	0.0039943
$\frac{1}{8\sqrt{15}}\ell^3$	4.0344×10^{-6}	3.2275×10^{-5}	0.0040344
β_o	0.0097001	0.019365	0.097070
$\frac{3}{4\sqrt{15}}\ell$	0.0096825	0.019365	0.096825

Case E - $\eta = \eta' = 0$ at one end of dipole.

This is the case of the bending magnet at either end of a zero-dispersion straight section. Let the zero-dispersion end be at $s = -s_1$ and the other end of the magnet be at $s = \ell - s_1$. We want to minimize $k^2\langle H\rangle$ with respect to s_1 and β_o. First, we have

$$k\eta_o' = \sin s_1 \ , \quad 1 - k\eta_o = \cos s_1$$

From Eq. (9) we get

$$k^2 H = \beta_o(\sin s_1 + \sin s)^2 + \frac{1}{\beta_o}(\cos s_1 - \cos s)^2 \tag{25}$$

Averaging over the magnet we have

$$k^2\langle H\rangle = \frac{k^2}{\ell}\int_{-s_1}^{\ell - s_1} H\, ds = \beta_o(E+F) + \frac{1}{\beta_o}(E-F) \tag{26}$$

where

$$\begin{cases} E = 1 - S_1 \\ F = \frac{1}{2}\left[(2C_1 - C_2)\sin 2s_1 + (2S_1 - S_2 - 1)\cos 2s_1\right] \end{cases} \tag{27}$$

To minimize $k^2\langle H\rangle$, we want to maximize F. This gives

$$\tan 2s_1 = \frac{2C_1 - C_2}{2S_1 - S_2 - 1} \tag{28}$$

which when expanded gives the approximation

$$s_1 \cong \frac{3}{8}\,\ell. \tag{29}$$

We, then, obtain the minimum value

$$k^2\langle H\rangle_{min} = 2\sqrt{E^2 - F^2} \cong \frac{1}{4\sqrt{15}}\,\ell^3 \tag{30}$$

at

$$\beta_o = \sqrt{\frac{E-F}{E+F}} \cong \frac{3}{8\sqrt{15}}\,\ell \tag{31}$$

The values of β at the zero-dispersion end, β_{e1}, and at the other end, β_{e2}, are

$$\begin{cases} \beta_{e1} = \dfrac{E - F\cos 2s_1}{\sqrt{E^2 - F^2}} \cong \dfrac{6}{\sqrt{15}}\,\ell \\ \beta_{e2} = \dfrac{E - F\cos 2(\ell - s_1)}{\sqrt{E^2 - F^2}} \cong \dfrac{16}{\sqrt{15}}\,\ell \end{cases} \tag{32}$$

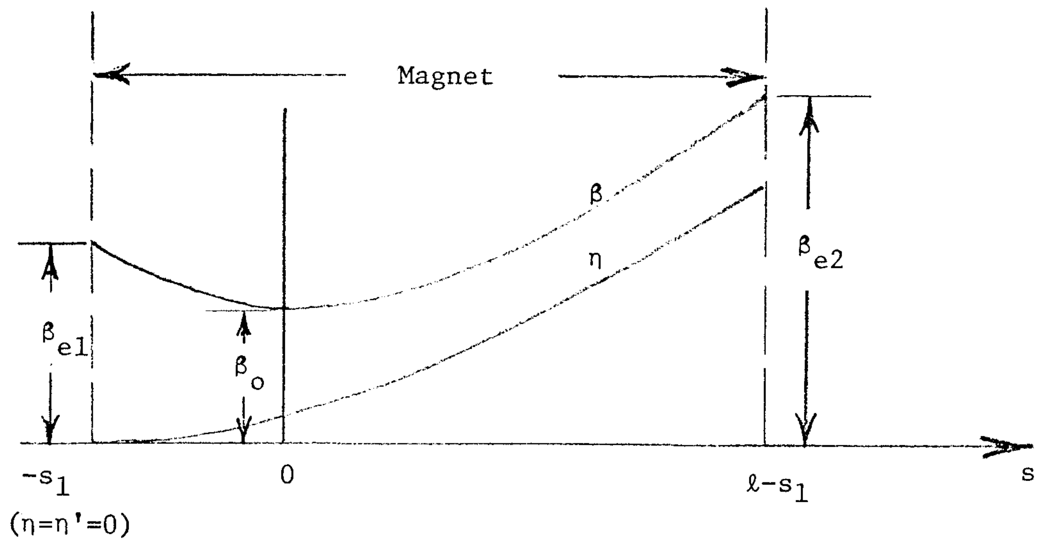

Figure 2. The betatron and dispersion functions of a bending magnet at an end of a zero-dispersion straight section.

The comparison between exact and approximate values are as follows:

	$\ell = 0.05$	$\ell = 0.10$	$\ell = 0.50$
$k^2\langle H\rangle_{min}$	8.0562×10^{-6}	6.4520×10^{-5}	0.0079637
$\frac{1}{4\sqrt{15}}\ell^3$	8.0687×10^{-6}	6.4550×10^{-5}	0.0080687
β_o	0.0048344	0.0096837	0.048497
$\frac{3}{8\sqrt{15}}\ell$	0.0048412	0.0096824	0.048412
s_1	0.018750	0.037498	0.18730
$\frac{3}{8}\ell$	0.018750	0.037500	0.18750

<u>Case E1</u> - $\eta = \eta' = 0$ at one end and $s_1 = \ell/2$.

For reasons of not wanting the β value to get too large, one may not be able to obtain the optimal value of $s_1 \simeq \frac{3}{8}\ell$. Thus, we look at the case when the beam waist (minimum value β_o) is at the midpoint ($s_1 = \ell/2$) of the magnet. In this case,

$$\begin{cases} E_1 = 1-S_1 \\ F_1 = \frac{1}{2}[(2C_1-C_2)\sin\ell + (2S_1-S_2-1)\cos\ell] \end{cases} \tag{33}$$

and we have

$$k^2\langle H\rangle_{min} = 2\sqrt{E_1^2-F_1^2} \simeq \frac{1}{3\sqrt{10}}\ell^3 \tag{34}$$

at

$$\beta_o = \sqrt{\frac{E_1-F_1}{E_1+F_1}} \simeq \frac{1}{2\sqrt{10}}\ell \tag{35}$$

The end value of β is

$$\beta_e = \frac{E_1-F_1\cos\ell}{\sqrt{E_1^2-F_1^2}} \simeq \frac{11}{2\sqrt{10}}\ell \tag{36}$$

The comparison between exact and approximate values looks as follows:

	$\ell = 0.05$	$\ell = 0.10$	$\ell = 0.50$
$k^2\langle H\rangle_{min}$	1.3167×10^{-5}	1.0535×10^{-4}	0.012976
$\frac{1}{3\sqrt{10}}\ell^3$	1.3176×10^{-5}	1.0541×10^{-4}	0.013176
β_o	0.0079018	0.015814	0.079329
$\frac{1}{2\sqrt{10}}\ell$	0.0079057	0.015811	0.079057

<u>Case E2</u> - $\eta = \eta' = 0$ at one end and $s_1 = 0$.

This is another case to check the sensitivity of the optimal value of s_1 given by Eq. (28). In this case, the beam waist (minimum value β_o) is at the zero-dispersion end. Inserting $s_1 = 0$ in Eq. (27) we get

$$\begin{cases} E_2 = 1 - S_1 \\ F_2 = \frac{1}{2}(2S_1 - S_2 - 1) \end{cases} \tag{37}$$

and the minimum value

$$k^2\langle H\rangle_{min} = 2\sqrt{E_2^2 - F_2^2} \cong \frac{1}{\sqrt{15}}\,\ell^3 \tag{38}$$

at

$$\beta_o = \sqrt{\frac{E_2 - F_2}{E_2 + F_2}} \cong \frac{3}{2\sqrt{15}}\,\ell \tag{39}$$

The β value at the other end is

$$\beta_{e2} = \frac{E_2 - F_2 \cos 2\ell}{\sqrt{E_2^2 - F_2^2}} \cong \frac{23}{2\sqrt{15}}\,\ell \tag{40}$$

The comparison table is the following:

	$\ell = 0.05$	$\ell = 0.10$	$\ell = 0.50$
$k^2\langle H\rangle_{min}$	3.2259×10^{-5}	2.5779×10^{-4}	0.031012
$\frac{2}{\sqrt{15}}\,\ell^3$	3.2275×10^{-5}	2.5820×10^{-4}	0.032275
β_o	0.019365	0.038746	0.19562
$\frac{1}{\sqrt{15}}\,\ell$	0.019365	0.038730	0.19365

C. Discussions

(1) In Case A we minimized the equilibrium emittance with respect to the four parameters s_o, β_o; η_o, η_o' two each specifying the betatron and the dispersion functions, respectively. Therefore, for a pure dipole (k=1) with a given bend angle $\theta \equiv \ell/\rho$, Case A represents the absolute minimum emittance. The only way to further reduce the emittance is by introducing a field gradient ($k \neq 1$). This dependence on k will be studied below.

In Case E we fixed both η_o and η_o' at the non-optimal values of $\eta = \eta' = 0$ at one end of the magnet. This case corresponds to that for bending magnets at the ends of zero-dispersion straight sections. If there are N_O zero-dispersion straight sections in the lattice, there must be $N_E = 2N_O$ Case E-type bending magnets. The number, N_A, of Case A-type magnets is, however, undetermined.

(2) The non-zero optimal value of η_o given in Eq. (17) for Case A is a little surprising. But, because of the rather complex dependence of H on η_o and η_o', one can not really expect this optimal value of η_o to be transparently obvious. In Case A1, η_o was arbitrarily fixed at the value zero. Comparison of Cases A and A1 shows that going to $\eta_o = 0$ leads to only a 50% increase in emittance. For the lattice design, therefore, any η_o between zero and the optimal value is acceptable.

(3) In Cases E1 and E2, we fixed s_1 at arbitrary non-optimal values to get a measure of the sensitivity of the emittance to s_1. Comparison of Cases E, E1, and E2 shows that ε_{min} is sensitive to s_1, the location of minimum β_o. In designing a lattice, it is therefore worthwhile to try to get, at least, close to the optimal value of s_1 as given by Eq. (28).

(4) In real units, the approximate formulas derived above can be summarized as follows:

$$\begin{cases} \frac{1}{\rho}\langle H\rangle_{min} \simeq K_i\ \theta^3 \\ \quad \text{or} \quad \varepsilon_{min} \simeq K_i \frac{C_q}{J_x} \gamma^2 \theta^3 \\ \beta_o \cong L_i \ell \qquad \text{and} \qquad \beta_e \cong M_i \ell \end{cases} \tag{41}$$

and

$$\begin{cases} \eta_o \cong \frac{1}{24}\ \theta\ell \qquad \text{(Case A)} \\ s_1 \cong \frac{3}{8}\ \ell \qquad \text{(Case E)} \end{cases} \tag{42}$$

where $\theta \equiv \ell/\rho$ is the bend angle in the magnet and the constants K_i, L_i, and M_i are, for the various cases,

$$\begin{array}{lll} K_A = \frac{1}{12\sqrt{15}} & L_A = \frac{1}{2\sqrt{15}} & M_A = \frac{8}{\sqrt{15}} \\ K_{A1} = \frac{1}{8\sqrt{15}} & L_{A1} = \frac{3}{4\sqrt{15}} & M_{A1} = \frac{23}{4\sqrt{15}} \\ K_E = \frac{1}{4\sqrt{15}} & L_E = \frac{3}{8\sqrt{15}} & M_E = \frac{6}{\sqrt{15}}\ ,\ \frac{16}{\sqrt{15}} \\ K_{E1} = \frac{1}{3\sqrt{10}} & L_{E1} = \frac{1}{2\sqrt{10}} & M_{E1} = \frac{11}{2\sqrt{10}} \\ K_{E2} = \frac{1}{\sqrt{15}} & L_{E2} = \frac{3}{2\sqrt{15}} & M_{E2} = \frac{23}{2\sqrt{15}} \end{array} \tag{43}$$

The θ^3 dependence of ε, the $\theta\ell$ dependence η, and the ℓ dependence of β are all expected. The dependences on all other parameters are through the constants K, L, and M. Ratios between different values of these constants give the sensitivity of optimization. For example, $K_{A1}/K_A = 3/2$, and

$K_{E1}/K_E = 4/\sqrt{6} = 1.63$ show that cases A1 and E1, although not optimal, are acceptable; whereas $K_{E2}/K_E = 4$ allows that Case E2 is a little too far from optimum.

(5) The approximate Eqs. (41) and (42) are all independent of the field gradient parameter k. Thus, for $k\theta = k\ell/\rho < 1$ when the approximation is good, gradients in magnets have little effect and one may as well use a separated function lattice with pure dipoles, quadrupoles, and sextupoles. One expects that for a focusing gradient in the bending magnet ($k > 1$) the emittance will be reduced and that for $k\theta > 1$ the reduction will be noticeable. From the first of Eqs. (41) and the exact formulas such as Eq. (15), we can write e.g., K_A as a function of $k\theta$

$$K_A(k\theta) = \frac{2\sqrt{A^2-B^2}}{(k\theta)^3} \tag{44}$$

and similarly for all other K_i's. The functions $K_A(k\theta)$ and $K_E(k\theta)$ are given for various values of $k\theta$ in the following table.

$k\theta$	$K_A(k\theta)$	$K_E(k\theta)$
0.1	0.0215	0.0645
0.5	0.0213	0.0637
1.0	0.0206	0.0612
1.5	0.0195	0.0573
2.0	0.0181	0.0522
3.0	0.0146	0.0397
4.0	0.0107	0.0268
5.0	0.0073	0.0158
10.0	0.00096	0.00173

These values of K_A and K_E should be compared to the limiting values of $K_A = \frac{1}{12\sqrt{15}} = 0.0215$ and $K_E = \frac{1}{4\sqrt{15}} = 0.0645$ given in Eqs. (43) for $k\theta \to 0$.

This table shows that the emittance is not significantly reduced, say, by a factor of more than 1.5 until $k\theta$ is larger than ~3.

(6) The energy scaling of the lattice is the following. One generally wants to use the same bending field strength and would like to keep the normalized emittance $\varepsilon_n \equiv \gamma\varepsilon \propto \gamma^3\theta^3$ fixed. These conditions lead immediately to the scaling relations:

Bending field	$B \propto \gamma^0$ (fixed)
Magnet length	$\ell \propto \gamma^0$ (fixed)
Bending radius	$\rho \propto \gamma$
Bend angle	$\theta \propto \gamma^{-1}$
Number of magnets	$N \propto \gamma$
Minimum significant k	$k \propto \theta^{-1} \propto \gamma$
Minimum significant gradient	$B'/B \propto \gamma$

For high energy machines, the useful gradient can get excessively large. For a low energy machine such as the 1 GeV TLS with $\theta = \frac{\pi}{6}$, to get $k\theta \cong 3$, one needs $k \cong 6$. At $\rho \cong 3$ m, this corresponds to $\frac{B'}{B} \cong 12\ \mathrm{m}^{-1} = 12\%$ per cm which is attainable. For the 6-GeV machine, however, with $\theta \cong \frac{\pi}{32}$, to get $k\theta \cong 3$, one needs $k \cong 30$ or $\rho\frac{B'}{B} \cong 900$. Even with a very large $\rho \cong 30$ m, the significant gradient is still $\frac{B'}{B} = 30\%$ per cm which is very difficult to achieve. At a maximum attainable gradient of roughly half that value, one gets $k\theta \cong 2$ or a reduction in emittance by a factor of only ~1.2.

In general, since a factor 2 reduction in emittance can readily be achieved by reducing θ by a factor $2^{1/3} = 1.26$ and hence, increasing the number of bending magnets by 1.26; for reducing emittance it is hardly ever profitable to bother with gradient magnets. On the other hand, it is possible that combined function magnets may have other applications.

D. Procedure for Designing a Minimum Emittance Lattice

Step 1. Choose a separated function lattice. Gradient magnets may be included only for reasons other than to minimize the beam emittance.

Step 2. Based on the desired emittance, choose the dipole bend angle $\theta \equiv \ell/\rho$, thence the corresponding total number $N_D = \frac{2\pi\rho}{\ell}$ of dipoles. To start, one may assume that all dipoles are identical. Higher N_D gives lower emittance but increases the number of magnets (especially quadrupoles) and hence the circumference and the cost.

Step 3. Determine the number of zero-dispersion straight sections, N_O, desired. The number of Case E type of dipoles is, then, $N_E = 2N_O$ and the number of Case A type of dipoles is $N_A = N_D - N_E$. One can now compute the theoretical minimum emittance obtainable with these dipoles from Eqs. (1), (19) and (30). This theoretical minimum should be at least a factor 2 smaller than the desired emittance because when one gets to the later steps, it is unlikely that one can attain and then maintain optimum values for all the parameters. The lattice now looks like that shown in Fig. 3.

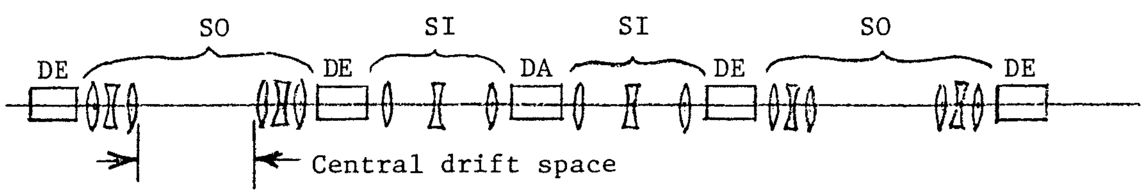

Figure 3. Lattice of dipoles and straight section where DA = Case A type dipole, DE = Case E type dipoles, SO = zero dispersion straight section, SI = non-zero dispersion straight section. Lenses are quadrupoles.

Quadrupoles are placed in the straight sections to produce the desired betatron and dispersion functions in the dipoles.

We have assumed so far that all dipoles have the same length. But the values of K for the A and E types of dipole differ by a factor 3. To really minimize the emittance the dipole lengths should be in the ratio $\frac{DA}{DE} = (3)^{1/3} = 1.44$.

Step 3. The zero-dispersion straight sections are used for insertion devices. Depending on the device, specific requirement is made on the length of the central clear drift space (6 m or longer for undulator, 2-3 m for wiggler). The beam sizes, hence the betatron functions in these devices should be adjustable. Thus, a minimum of 3 quadrupoles must be placed between the central drift space and the end dipole to produce the desired β-function both in the dipole and in the central drift space. It is best to have 4 quadrupoles to provide greater flexibility. Generally a mixture of SO-straights with different central drift lengths is desired. To keep the β values from getting too large, the polarities of the quadrupoles are likely those shown in Fig. 3.

Step 4. The non-zero straight sections have no special application. It is used only when one does not need all N_D zero-dispersion straight sections and wants to reduce the ring circumference, hence the cost, by making $N_D - N_O$ straight sections the non-zero dispersion type. The length of this type of straight section could be much smaller. For either type of straight section, the phase advance between the two dipoles at the ends is about 180° (exactly 180° between midpoints of two A-type dipoles). This can be produced by a single focusing quadrupole as in the Chasman-Green lattice, but three (or four) quadrupoles would be better. The polarities of the quadrupoles should be as shown in Fig. 3 in order to keep the β value not too high in the straight section. With the quadrupoles in each straight section so arranged, one now has the entire linear lattice. The quadrupoles will have to be adjusted slightly to put the betatron tunes ν_x and ν_y onto good values. This will presumably increase the emittance slightly from the minimum value.

Step 5. One is now ready to place the sextupoles. A minimum of two sets of sextupoles are required: both at high η, but one at high horizontal β and the other at high vertical β locations. It is likely that a third set located in zero-dispersion straight sections is needed to compensate for the undesirable harmonic non-linear effects of the first two sets and to obtain a sufficiently large dynamic aperture.

This "minimum emittance lattice" with a large number of quadrupoles and 180° phase advance between dipoles will need rather strong sextupoles to compensate for the rather large chromaticities. Thus, the dynamic aperture is very sensitive to the tuning of the quadrupoles and the sextupoles. After the rough design is complete, one then has to carry out rather extensive detailed tuning to obtain a sufficiently large dynamic aperture and to ensure that it is not overly sensitive to the chosen operating point.

Step 6. Finally, one wants to find out the range of tuning of the β-functions in the drift spaces of zero-dispersion straight sections, SO. This is done by adjusting the two sets of 3 (or 4) quadrupoles at the ends of the straight section. With each tuning of the quadrupoles, one has to recompensate for the chromaticities by re-adjusting the sextupoles and re-investigate the adequacy of the dynamic aperture. Of course, it would be impossible to study all possible tuning combinations. Some one dozen of typical combinations will do. During operation when a new combination of β-tunings in the straight sections is desired, this new tune must be investigated immediately for the adequacy of the dynamic aperture by computer simulation before it is set up on the storage ring for the following operating period.