Chromaticity Correction and Betafunction Distortion

The required value of the chromaticity is obtained by introducing sextupole magnets in the dispersive straight sections

$$\frac{p}{v} \frac{\Delta v}{\Delta p} = - \frac{1}{4\pi v} \int \beta(K - S\eta) ds,$$

where $K = \frac{1}{B_p} \frac{dB}{dx}$ is the focusing strength of the lattice quadrupoles, $S = \frac{1}{B_p} \frac{d^2B}{dx^2}$ the strength of the correction sextupoles and $\eta$ is the dispersion function. About one half of the quadrupoles are located in dispersion-free straight sections. Furthermore, the natural chromaticity of the low-emittance lattice is large and one will have large harmonic components in the Fourier series expansion of $\beta(K - S\eta)$. Since the beta functions depend on the focusing strength, these Fourier components will effect the beta functions of the off-momentum particles. Analogous to the definition of chromaticity, one can write the momentum spread dependence of the beta function in the form $\frac{p}{\beta} \frac{d\beta}{dp}$. Using the Twiss parameter relations

$$\frac{d\beta}{ds} = -2 \alpha$$
$$\frac{d\alpha}{ds} = K\beta - \frac{1+\alpha^2}{\beta}$$

one can show that this quantity $\frac{p}{\beta} \frac{d\beta}{dp}$ satisfies the equation

$$\frac{d^2}{d\phi^2} \left( \frac{p}{\beta} \frac{d\beta}{dp} \right) + 4 \left( \frac{p}{\beta} \frac{dB}{dp} \right) = -2\beta^2 \frac{p}{dp} \frac{dK}{dp},$$
where $\phi = \int \frac{ds}{\beta}$ is the betatron phase advance. Setting $\frac{dK}{dp} = (K - S)$ and replacing $d\phi$ by $vd\psi$, one obtains

$$\frac{d^2}{d\psi^2} \left( \frac{d}{d\psi} \right) + (2n)^2 \left( \frac{d}{d\psi} \frac{d\beta}{dp} \right) = -2n^2 \beta^2 (K - S).$$

Expansion of $v\beta^2(K - S)$ in the Fourier series

$$v\beta^2(K - S) = \sum a_n e^{-inN\psi},$$

where $N$ is the number of superperiods,

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} v\beta^2(K - S)e^{-inN\psi} d\psi$$

$$= \frac{1}{2\pi} \int_0^{L} (K - S) e^{-inN\psi(s)} ds,$$

and $L$ is the length of the superperiod, gives the periodic solution

$$\frac{d}{dp} \frac{d\beta}{dp} = \sum \frac{2n a_n}{n^2 - (2n)^2} e^{-inN\psi(s)}. \quad (1)$$

The above equations apply for both transverse motion, with $K = K_x$, $\nu = \nu_x$, $\beta = \beta_x$, $\psi = \phi_x/\nu_x$ for the horizontal motion, and $K = K_y$, $\nu = \nu_y$, $\beta = \beta_y$, $\psi = \phi_y/\nu_y$ for the vertical motion. From Eq. (1) one sees that the beta function distortion is most sensitive to the Fourier components whose order is close to $2n$. Therefore, the choice of the locations and strengths of the sextupoles should not only be determined by maximizing the stability limits but also by minimizing the quantities
\[ a_m = \frac{1}{2\pi} \int_0^L \beta_x (K_x - S_n) e^{-i m_x s \psi_x(s)} ds, \]

with \( m_x \) close to \( 2\nu_x \),

and

\[ a_m = \frac{1}{2\pi} \int_0^L \beta_y (K_y - S_n) e^{-i m_y s \psi_y(s)} ds, \]

with \( m_y \) close to \( 2\nu_y \).